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5530. (J. W. L. Glaisher, F.R.S.)—There being $(n+1)^2$ quantities

$$\begin{array}{llll} a_1a_1, & a_1a_2, & \ldots, & a_1a_n, \quad a_1a_{n+1}, \\ a_2a_1, & a_2a_2, & \ldots, & a_2a_n, \quad a_2a_{n+1}, \\ \ldots & \ldots & \ldots & \ldots \quad \ldots \\ a_na_1, & a_na_2, & \ldots, & a_na_n, \quad a_na_{n+1}, \\ a_{n+1}a_1, & a_{n+1}a_2, & \ldots, & a_{n+1}a_n, \quad a_{n+1}a_{n+1}, \end{array}$$

of which only $\frac{1}{2}(n+1)(n+2)$ are different, as it is supposed that $a_pa_q = a_qa_p$, thence prove that

$$\begin{vmatrix} a_1a_1, & a_1a_2, & \ldots, & a_1a_n, & a_1a_{n+1} \\ a_2a_1, & a_2a_2, & \ldots, & a_2a_n, & a_2a_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_na_1, & a_na_2, & \ldots, & a_na_n, & a_na_{n+1} \\ a_{n+1}a_1, & a_{n+1}a_2, & \ldots, & a_{n+1}a_n, & a_{n+1}a_{n+1} \end{vmatrix} + \begin{vmatrix} a_1a_1, & a_1a_2, & \ldots, & a_1a_n \\ a_2a_1, & a_2a_2, & \ldots, & a_2a_n \\ \ldots & \ldots & \ldots & \ldots \\ a_na_1, & a_na_2, & \ldots, & a_na_n \end{vmatrix}$$

$$= a_{n+1}a_{n+1} - \frac{(a_1a_{n+1})^2}{a_1a_1} - \frac{(a_2a_{n+1}\,.\,1)^2}{a_2a_2\,.\,1} - \frac{(a_3a_{n+1}\,.\,2)^2}{a_3a_3\,.\,2} - \ldots - \frac{[a_na_{n+1}\,(n-1)]^2}{a_na_n\,(n-1)},$$

where $a_2a_{n+1}\,.\,1 = a_2a_{n+1} - \frac{a_2a_1\,.\,a_{n+1}a_1}{a_1a_1}$, $a_2a_2\,.\,1 = a_2a_2 - \frac{a_2a_1\,.\,a_2a_1}{a_1a_1}$;

and, generally, $a_pa_q\,.\,i = a_pa_q\,.\,i-1 - \frac{[a_pa_i\,(i-1)][a_qa_i\,(i-1)]}{a_ia_i\,(i-1)}$ 95

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$$\frac{AA'.\sin^2\alpha + \ldots - (BC'+B'C)(\cos\alpha - \cos\beta\cos\gamma) - \ldots}{[A^2\sin^2\alpha + \ldots - 2BC\,(\cos\alpha - \cos\beta\cos\gamma) - \ldots]^{\frac{1}{2}} \times [A'^2\sin^2\alpha + \ldots - 2B'C'\,(\cos\alpha - \cos\beta\cos\gamma)]^{\frac{1}{2}}};$$

and (2) the length of the perpendiculars from the point (xyz) on the plane $Ax+By+Cz+D=0$ is

$$(Ax+By+Cz+D)\left(\frac{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma}{A^2\sin^2\alpha+\ldots-2BC\,(\cos\alpha-\cos\beta\cos\gamma)}\right)^{\frac{1}{2}}$$ 79

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$S' = \frac{a}{m+1}\left\{\left(\cos\frac{m}{m+1}\phi\right)^{(1-m)/m} - 1\right\}$, where ϕ has its usual meaning

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12174. (Rev. T. C. Simmons, M.A.)—Three or more circular wafers of different sizes are fastened at random on the surface of a cylinder. Find the probability that they will all lie (1) on the same generating line, (2) on a generating line drawn at random; also (3) prove that (2) may be reduced to a particular case of (1) 77

12471. (Professor Sanjána, M.A.)—Investigate the following (or similar) simple formulæ for the sides of a triangle of rational area :—

$$mx^2+my^2, \quad m^2+x^2y^2, \quad (x^2-m)(y^2+m) \quad \text{..... } 101$$

12602. (M. Brierly.)—There are four numbers w, x, y, z; $wxy = 252$, $wxz = 756$, $xyz = 432$: find the numbers 88

12685. (R. Tucker, M.A.)—P(m_1), Q(m_2), R(m_3) are conormal points on a parabola, and pqr is the tangent triangle. PP_q, QQ_p, parallel to the tangents at Q, P, respectively, meet the curve in P_q, Q_p, and cut one another in R_r; P_p, Q_q are points obtained in a similar manner. PP_p, QQ_q meet in r_2, and so on; prove that

$$xy = -m_1m_2m_3, \ a^2 \quad \text{..... ..... (i.)}$$

is a hyperbola circumscribing the triangles pqr, $p_2q_2r_2$, and that the orthocentres of these triangles coincide, and, of course, lie on (i.) The circumcircles of these triangles cut the latus rectum in the same point, the join of which to the orthocentre is a diameter of (i.). The parallels through P, Q to the normals at Q, P intersect in r, and so on; then the orthocentre of $p_1q_1r_1$ is on (i.), as is also the centre of perspective of the triangles PQR, pqr 119

12904 (Professor Morley.)—The tangents at the points where any tangent of a hypocycloid of class 4 (the so-called four-cusped hypocycloid) meets the curve again meet on the cusp circle 73

13029. (Vve. F. Prime.)—Une circonférence, menée par le point de rebroussement d'une développée de parabole, coupe la courbe en quatre autres points que l'on joint au point de rebroussement. (1) Trouver le lieu du centre de la circonférence pour que le faisceau de ces quatre droites soit harmonique, et (2) démontrer que le centre des moyennes distances des quatre points d'intersection décrit une perpendiculaire à la tangente de rebroussement 97

13382. (Professor Nanson.)—If u, v are n-ary quadrics in $x_1, \ldots, x_n$, and $I_m = \int \ldots \int v^m dx_1, \ldots, dx_n$, the limits being given by $0 \leqslant u \leqslant 1$, u being definite and positive, prove that $\delta I_m = -(\frac{1}{2}n+m+1)\, I_{m+1}$, where

$$\delta = a'(d/da) + b'(d/db) + \ldots,$$

a, b, ... being the coefficients of u, and a', b', ... those of v. Hence express I_m in terms of the invariants of u, v, m being a positive integer 44

14055. (G. H. Hardy.)—Express

$$\int_0^{\frac{1}{2}\pi} \log \sin x \sqrt{(\sin x)}\, dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\log \sin x}{\sqrt{(\sin x)}}\, dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\log \sin x}{\sqrt{(1+\sin^2 x)}}\, dx,$$

$$\int_0^{\pi} \frac{dx}{(\sec \frac{1}{2}x + \tan \frac{1}{2}x)^m \sqrt{(\sin x)}} \quad (m \text{ a positive integer}),$$

$$\int_0^{\pi} \frac{\log (\sec \frac{1}{2}x + \tan \frac{1}{2}x)}{\sqrt{(\sin x)}}\, dx, \quad \text{and} \quad \int_0^{\infty} \frac{x}{\sqrt{(\sinh x)}}\, dx$$

as multiples of $[\Gamma(\frac{1}{4})]^{\pm 2}$; and prove that

$$\int_0^{\frac{1}{2}\pi} \sin x\, (\log \sin x)^2\, dx = (\log 2 - 1)^2 + 1 - \tfrac{1}{12}\pi^2.$$

[The last integral is given in Quest. 8423 (D. Edwardes); another proof is asked for] 41

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14059. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{3^2+1}+\frac{1}{7^2-1}+\frac{1}{17^2+1}+\frac{1}{41^2-1}+\ldots=\tfrac{1}{8},$$

$$\frac{1}{1^2+9}-\frac{1}{7^2+9}+\frac{1}{41^2+9}-\frac{1}{239^2+9}+\ldots=\tfrac{1}{12},$$

$$\frac{1}{3^2-1}+\frac{1}{11^2-1}+\frac{1}{41^2-1}+\frac{1}{153^2-1}+\ldots=1-\tfrac{1}{2}\sqrt{3}\ldots\ldots 69$$

14079. (Rev. T. Roach, M.A.)—Prove geometrically

$$\sin^{-1}\tfrac{4}{5}+\sin^{-1}\tfrac{5}{13}+\sin^{-1}\tfrac{16}{65}=\tfrac{1}{2}\pi\ldots\ldots 114$$

14097. (J. J. Barniville, B.A.)—Having $u_{n-1}+u_{n+1}=3u_n$, prove that

$$\frac{1}{10+11}+\frac{1}{13+11}+\frac{1}{29+11}+\frac{1}{74+11}+\ldots=\tfrac{2}{7}\frac{\sqrt{5}}{7+\sqrt{5}},$$

$$\frac{1}{2^2+1}+\frac{1}{3^2+1}+\frac{1}{7^2+1}+\frac{1}{18^2+1}+\ldots=\frac{\sqrt{5}+1}{10},$$

$$\frac{1}{1^2+9}-\frac{1}{4^2+9}+\frac{1}{11^2+9}-\frac{1}{29^2+9}+\ldots=\tfrac{1}{15},$$

$$\frac{1}{7+3}+\frac{1}{18-3}+\frac{1}{47+3}+\frac{1}{123-3}+\ldots=\tfrac{1}{6}\ldots\ldots 102$$

14405. (H. MacColl, B.A.)—To make sense of the implication, "If it is x that A and B are y, it is y that A and B are x," one of the three words, *true*, *certain*, *probable*, is taken at random and substituted for x; then one of the same three words is taken at random and substituted for y. What is the chance that the implication, thus rendered intelligible, will be a *formal certainty*? (I find *two-thirds*; but my reasoning involves one or two controvertible points, and I should be glad to see other solutions.) See *Reprint*, Vol. LXXIV., p. 36 42

14537. (R. Chartres.)—A point P is taken at random in a polygon ABCD ... of n sides. What is the probability that, if n other random points be taken, one shall lie in each of the triangles APB, BPC, &c.? 40

14542. (Professor E. Lemoine.)—Dans un triangle ABC soient G le barycentre; L, M, N les milieux de BC, CA, AB; l, m les milieux de AG, BG; D le point situé sur LN au tiers de LN à partir de N. Pour trouver le centre de l'hyperbole de Kiepert, il suffira de prendre le point I où lM recoupe le cercle des neuf points; de prendre le point H où ID recoupe ce cercle; Hm recoupera le même cercle au centre T cherché. 98

14575. (J. A. Third, M.A., D.Sc.)—If A′B′C′ be the first Brocard triangle of ABC, and D, E, F be the points where B′C′, C′A′, A′B′ meet BC, CA, AB respectively, show (1) that DA′, EB′, FC′ are tangents to the Brocard circle of ABC, and (2) that DA, EB, FC are tangents to the circumconic of ABC which passes through the Brocard points ... 79

14631. (Professor Sanjána, M.A.)—A, B, C being the angles of a triangle, prove that

$$\cos A\sin^3 A\sin^4(B-C)+\cos B\sin^3 B\sin^4(C-A)+\cos C\sin^3 C\sin^4(A-B)$$
$$=\sin A\sin B\sin C\left\{\sin^2(A-B)\sin^2(C-A)+\sin^2(B-C)\sin^2(A-B)\right.$$
$$\left.+\sin^2(C-A)\sin^2(B-C)\right\}$$
..... 3

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14642. (Robert W. D. Christie.)—Establish directly, by use of the quintic roots of minus unity, Dr. Reuschle's *Einfach Factor* of the prime 11, and, if possible, extend same to other primes..... 83

14652. (Professor E. Lemoine.)—(1) Le point Φ du triangle ABC, dont les coordonnées barycentriques sont $a^2b^2+a^2c^2-b^2c^2$, $b^2c^2+b^2a^2-c^2a^2$, $c^2a^2+c^2b^2-a^2b^2$, et qui jouit, d'ailleurs, de remarquables propriétés, est à l'intersection de la droite qui joint le milieu de la distance des points de Brocard au barycentre, et sur chacune des trois droites qui joignent le symétrique d'un sommet par rapport au milieu du côté opposé, au pied, sur le côté, de la symédiane partant du sommet opposé. (2) Soient A_1, B_1, C_1 les points où AΦ, BΦ, CΦ coupent BC, CA, AB ; les parallèles à AB et à AC menées par A_1 coupent respectivement AC et AB en μ et ν' ; les parallèles à BC et à BA menées par B_1 coupent respectivement BA et BC en ν et λ' ; les parallèles à CA et à CB menées par C_1 coupent respectivement BC et BA en λ et μ'. Cela posé, les points $\lambda, \mu, \nu, \lambda', \mu', \nu'$ sont concycliques 85

14655. (J. A. Third, M.A., D.Sc.)—From the vertices of a triangle ABC lines AX, BY, CZ are drawn to the opposite sides, such that the angles CAX, ABY, BCZ, taken the same way round, are together equal to 180° ; and AX′, BY′, CZ′ are drawn from the vertices to the opposite sides, parallel to CZ, AX, BY respectively. Show that the circum-centres of AXX′, BYY′, CZZ′ are collinear 30

14659. (D. Biddle.)—N being any odd number that is prime to 3 and 7, prove that $N^6 \equiv 1 \pmod{168}$ 76

14660. (R. Tucker, M.A.) — PQ is a focal chord of a parabola (focus S), and is also the major axis of an ellipse. Find the locus of the other focus and the equation to the ellipse. If this ellipse cuts the axis in K, K′, then $1/SK + 1/SK' = 1/a$, where $4a$ is the latus rectum of the parabola (and also of the ellipse). Show also that the envelope of the ellipse is a confocal parabola of latus rectum $8a$, and that of the auxiliary circle is (besides the directrix) the circle $x^2+y^2-ax = 2a^2$. If $\lambda = \cot\phi$ (ϕ angle made by chord with axis), then the point of contact is $x/a = -2(2\lambda^2-1)/(4\lambda^2+1)$, $y/a = -6\lambda/(4\lambda^2+1)$ 49

14665. (H. MacColl, B.A.) — Each letter denoting a statement, and α' being the denial of α, show that $(ax = ay)(a'x = a'z)$ and $\alpha(x = y) + \alpha'(x = z)$ are equivalent when α is a *constant* (x, y, z being unrestricted), but not necessarily when α is a *variable*. [*Def.* : A statement is a *constant* when it is *either certain or impossible* ; a *variable* when it is *neither certain nor impossible*] 60

14692. (Robert W. D. Christie.)—Find perfectly general expressions for the sum of any n integral squares equal to the sum of n others when the sums of their roots are equal, also $(n > 2)$: *e.g.*, $n = 3$,

$$(a+b+c)^2+(a+d+c)^2+(a+c-b-d)^2 = (a+c-b)^2+(a+c-d)^2+(a+b+c+d)^2$$

..... 40

14693. (J. J. Barniville, B.A.) — Having $u_n = u_{n-2} + u_{n-3}$, prove that

$$\frac{1}{2.4.5}+\frac{1}{3.5.7}+\frac{1}{4.7.9}+\frac{1}{5.9.12}+\ldots = \frac{1}{24},$$

$$\frac{1}{2.3.4.5.9}+\frac{1}{3.4.5.7.12}+\frac{1}{4.5.7.9.16}+\ldots = \frac{1}{840},$$

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$$\frac{2.4.12}{3.5.7.9}+\frac{3.5.16}{4.7.9.12}+\frac{4.7.21}{5.9.12.16}+\ldots=\frac{1}{3}+\frac{1}{4}-\frac{1}{7},$$

$$\frac{2^2}{1.3.5}+\frac{3^2}{1.4.7}+\frac{4^2}{1.5.9}+\frac{5^2}{2.7.12}+\ldots=\frac{41}{24},$$

$$\frac{1}{9^2-1^2}+\frac{2}{12^2-2^2}+\frac{2}{16^2-2^2}+\frac{3}{21^2-3^2}+\ldots=\frac{1}{16} \quad \ldots\ldots\ldots\ldots \quad 28$$

14695. (R. Knowles.)—Prove that the sums to r terms of the series

(1) $1-(m+1)+(2m+1)-(3m+1)\ldots\pm(mr-m+1)$,

(2) $1-(m+1)^2+(2m+1)^2-(3m+1)^2\ldots\pm(mr-m+1)^2$,

(3) $1-(m+1)^3+(2m+1)^3-(3m+1)^3\ldots\pm(mr-m+1)^3$

are respectively

$-\frac{1}{2}rm$, $\frac{1}{2}rm(m-rm-2)$, and $-\frac{1}{4}rm\left\{r(2r-3)m^2+6(r-1)m+6\right\}$,

when r is even,

and $\frac{1}{2}(mr-m+2)$, $\frac{1}{2}rm\left\{(rm-m+2)-2(m-1)\right\}$,

and $\frac{1}{4}\left\{(r-1)^2(2r+1)m^3+6r(r-1)m^2+6(r-1)m+4\right\}$,

when r is an odd number 58

14710. (Professor N. Bhattacharyya.)—From the centre of curvature at any point on the ellipse $(x^2/a^2)+(y^2/b^2)=1$ two other normals are drawn. Show that the envelope of the line joining their feet is the curve $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$ 51

14725. (Robert W. D. Christie.)—Prove the following theorems on circulators. P = any prime having an *even* period; H_1 = the number formed by the figures of the first half; H_2 = the number formed by the figures of the second half. Then

$$P-1=(H_2+1)/(H_1+1),\quad H_2(P-1)=(P-2)10^{\frac{1}{2}p}+H_1 \quad \ldots(1,2),$$

$$(H_2+1)/(H_1+1)=(10^{\frac{1}{2}p}-H_1)/(10^{\frac{1}{2}p}-H_2),\quad 10^{\frac{1}{2}p}-H_1=0 \bmod P-1 \quad (3,4),$$

$$H_1+H_2+2=0 \bmod P=10^{\frac{1}{2}p}+1 \quad \ldots\ldots\ldots\ldots(5),$$

$$(H_1+H_2+2)/(H_2-H_1)=P/(P-2) \quad \ldots\ldots\ldots\ldots(6),$$

$$P(H_1+1)=10^{\frac{1}{2}p}+1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and various others by combination.

Example of (1)— $P=7$, $\frac{1}{7}=142857$;

therefore $7-1=(857+1)/(142+1)=6$ 65, 84

14739. (J. A. Third, M.A., D.Sc.)—M and N are points on the sides CA, AB of a triangle, such that $CA.CM=BA.BN$. M′ and N′ are the images of M and N with respect to BC. Through M′ and N′ parallels are drawn to CA, AB respectively, intersecting in P. AP meets BC in D. The connector of A with the circum-centre of ABC meets BC in E. Show that BD = CE ..... 68.

14743. (Professor N. Bhattacharyya.)—Find the envelope of the chord of intersection of the ellipse $x^2/a^2+y^2/b^2=1$, and the circle which touches it and passes through its centre. Show that the first positive pedal of the envelope is the ellipse $x^2/b^2+y^2/a^2=(1-e^2)/e^4$ 39

14748. (James Blaikie, M.A.)—If a tetrahedron have its opposite edges equal, its four escribed spheres are equal, and the radius of an escribed sphere is equal to twice the radius of the inscribed sphere ... 88.

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14751. (H. MacColl, B.A.)—The chance that A is true (assuming nothing as to B) is a; the chance that B is true (assuming nothing as to A) is b; and the chance that A is true, assuming B true, is k times the chance that B is false, assuming A false. Show that the dependence of A upon B is $(1-a)(k-kb-a)/(1-a-kb)$, except when the numerator and denominator of this fraction both vanish. Discuss this exceptional case. (See *Reprint*, Vol. LXXII., pp. 79, 80) 29

14761. (A. F. van der Heyden, B.A.)—Let I, I_1, I_2, I_3 denote the in- and ex-centres of a triangle. Given the circum-centres of the triangles II_2I_3, II_3I_1, and of the original triangle, construct the triangle 62

14764. (Professor Neuberg.) — Construire un triangle ABC, étant donnés les côtés indéfinis AB, AC et le centre K des symédianes ... 78

14765. (Professor Jan de Vries.)—The equation

$$a_{11}x^2+2a_{12}xy+a_{22}y^2+b_1x+b_2y=0$$

being referred to rectangular axes, the axes are supposed to turn about the origin. Prove that the function $I=(a_{11}-a_{22})\,b_1b_2-a_{12}(b_1^2-b_2^2)$ does not undergo change of form. What is the proper geometric meaning of $I=0$? 107

14776. (J. Blaikie, M.A.) — Show that any number $10a+b$ is a multiple of $10n-1$, if $a+nb$ be a multiple of $10n-1$. *E.g.*,

63 is a multiple of 9, since $6+3$ is a multiple of 9;
351 ,, 39, ,, $35+4$,, 39;
1157 ,, 89, ,, $115+63$,, 89 ... 45

14800. (R. F. Davis, M.A.)—Prove that the equations $xy=ac+bu$, $yz=bd+cv$, $zu=ce+dx$, $uv=da+ey$, $vx=eb+az$ are not independent, and that, given any three, the other two follow 108

14801. (R. F. Muirhead, M.A., B.Sc.)—Show that, if the sum of any number of quantities = 0, then the sum of their cubes is three times the sum of their products taken three at a time..... 91

14803. (H. D. Drury, M.A.)—C is a given circle, APB a given straight line: to find a point P in AB so that, when tangents PT, PT′ are drawn to C, $\angle APT = 2BPT'$ 68

14820. (Professor Langhorne Orchard, M.A., B.Sc.) — Find the product of the two series

$0+24+216+960+3000+\ldots$, to n terms,
$2+40+270+1088+3250+\ldots$, to n terms 92

14831. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{\sqrt{5}+3}+\frac{1}{2\sqrt{5}+3}+\frac{1}{5\sqrt{5}+3}+\frac{1}{13\sqrt{5}+3}+\ldots=\frac{1}{\sqrt{5}},$$

$$\frac{1}{\sqrt{6}+4}+\frac{1}{3\sqrt{6}+4}+\frac{1}{11\sqrt{6}+4}+\frac{1}{41\sqrt{6}+4}+\ldots=\frac{1}{\sqrt{12}},$$

$$\frac{1}{\sqrt{7}+5}+\frac{1}{4\sqrt{7}+5}+\frac{1}{19\sqrt{7}+5}+\frac{1}{91\sqrt{7}+5}+\ldots=\frac{1}{\sqrt{21}},$$

$$\frac{1}{\sqrt{2}+3}+\frac{1}{5\sqrt{2}+3}+\frac{1}{29\sqrt{2}+3}+\frac{1}{169\sqrt{2}+3}+\ldots=\frac{1}{2\sqrt{2}},$$

$$\frac{1}{\sqrt{3}+5}+\frac{1}{9\sqrt{3}+5}+\frac{1}{89\sqrt{3}+5}+\frac{1}{881\sqrt{3}+5}+\ldots=\frac{1}{2\sqrt{6}},$$

$$\frac{1}{2+7}+\frac{1}{26+7}+\frac{1}{362+7}+\frac{1}{5042+7}+\ldots=\frac{1}{4\sqrt{3}} \quad \ldots\ldots 56$$

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14839. (Professor Neuberg.) — Dans une parabole on mène la symétrique de la normale par rapport à l'ordonnée. Trouver l'enveloppe de cette droite..... 86

14843. (Professor Umes Chandra Ghosh.)—If K be the symmedian point of a triangle ABC, and KA, KB, KC produced cut the circumcircle of the triangle ABC at A_1, B_1, C_1 respectively, and if π_1, π_2, π_3 be the lengths of the perpendiculars from K on B_1C_1, C_1A_1, A_1B_1, prove that $1/a^2+1/b^2+1/c^2 = \frac{1}{4}(\tan^2\omega)(1/\pi_1^2+1/\pi_2^2+1/\pi_3^2)$, where ω is the Brocard angle of the triangle ABC 67

14849. (Lt.-Col. Allan Cunningham, R.E.) — Resolve into *prime* factors $N = (11^{33}+1)$ 118

14854. (Rev. T. Wiggins, B.A.)—Take a point P within a triangle ABC. Denote the lines PA, PB, PC by x, y, z, and the angles PAC, PBA, PCB by α, β, γ respectively; show that (1)

$$\frac{ax}{\sin(C-\gamma+\beta)} = \frac{by}{\sin(A-\alpha+\gamma)} = \frac{cz}{\sin(B-\beta+\alpha)};$$

(2) if $C-\gamma+\beta = 90°$,

then $a^2x^2 = b^2y^2+c^2z^2$ 34

14878. (J. J. Barniville, B.A.)—Having $u_n+u_{n+3} = 3u_{n+1}+3u_{n+2}$, prove that

$$\frac{1}{12}+\frac{1}{7}+\frac{1}{60}+\frac{1}{189}+\frac{1}{740}+\ldots = \frac{1}{4},$$

$$\frac{1}{24}+\frac{1}{9}+\frac{1}{104}+\frac{1}{315}+\frac{1}{1248}+\ldots = \frac{1}{6},$$

$$\frac{1}{40}+\frac{1}{11}+\frac{1}{160}+\frac{1}{473}+\frac{1}{1888}+\ldots = \frac{1}{8},$$

$$\frac{1}{15}+\frac{1}{12}+\frac{1}{85}+\frac{1}{276}+\frac{1}{1071}+\ldots = \frac{1}{6},$$

$$\frac{1}{21}+\frac{1}{36}+\frac{1}{175}+\frac{1}{612}+\frac{1}{2325}+\ldots = \frac{1}{12},$$

$$\frac{1}{20}+\frac{1}{39}+\frac{1}{180}+\frac{1}{637}+\frac{1}{2412}+\ldots = \frac{1}{12}$$ 82

14888. (D. Biddle.)—ABC being a plane triangle having $a > b > c$, produce BC to D, BA to E, CA to F; and prove that, if $CD = ab/(a-b)$, $BE = ac/(a-c)$, and $AF = bc/(b-c)$, then D, E, F are collinear 115

14891. (A. M. Nesbitt, M.A.)—ABC is a triangle, O any point. Tangents OP, OQ are drawn to the in-circle or one of the ex-circles. Show that OP touches the conic OABCQ, and OQ touches the conic OABCP 53

14896. (Salutation.)—In a circle (of radius = unity) is a chord of unknown length, and in the chord a point joined to the centre by a straight line which forms with the radius drawn to the nearest extremity of the chord an angle $= \alpha$. The projection upon this radius of the aforesaid point is at a distance from the centre $= 2\cos^2\alpha - \cos\alpha$. If $\cos\alpha = \frac{1}{4}\pi$, prove that the chord subtends an angle whose cosine $= -\cdot4183023\ldots$ 55

14906. (Professor Neuberg.)—A′, B′, C′ étant les pieds des hauteurs du triangle ABC, soient A_1, B_1, C_1 les symétriques des sommets A, B, C respectivement par rapport aux points A′, B′, C′. Calculer le rapport des aires des triangles $A_1B_1C_1$, ABC 52

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14913. (R. Tucker, M.A.)—O_a is the centre of the circle touching the circum-circle internally and also the sides AB, AC of the triangle ABC. Prove that

$$(1)\ \rho_a \cos^2\tfrac{1}{2}A = \rho_b \cos^2\tfrac{1}{2}B = \rho_c \cos^2\tfrac{1}{2}C = r,$$

$$(2)\ \Sigma OO_a = 3R - r\left\{1 + (\Sigma \tan\tfrac{1}{2}A)^2\right\},$$

where ρ_a, ρ_b, ρ_c are the respective radii of the touching circles 97

14917. (A. M. Nesbitt, M.A.)—Three conics S_1, S_2, S_3 have a common point O. The remaining intersections of S_2S_3 are the vertices of a triangle Δ_1, those of S_3S_1 of a triangle Δ_2, and those of S_1S_2 of a triangle Δ_3. Prove that the nine sides of these three triangles all touch the same conic U. Or, reciprocally: Given three parabolas, their common tangents, taking the curves in pairs, will form three triangles (one for each pair) whose nine vertices lie on a conic 105

14920. (Aletrop.)—Généralisation du théorème de Fermat. Démontrer, *sans se servir d'aucune interprétation géométrique*, que l'expression

$$\Sigma(x) = x^n - \Sigma x^{n/a} + \Sigma x^{n/ab} - \Sigma x^{n/abc} + \ldots,$$

où a, b, c, sont les facteurs premiers de n, est divisible par n, quels que soient n et x. [M. Kantor, dans sa mémoire:—"Wie viele cyclische Gruppen gibt es in einer quadratischen Transformation der Ebene?" (*Annali di Matematica pura ed applicata*, Tom x., p. 64), trouva le théorème de l'énoncé pour le cas $x = 2$; et M. Picquet le démontra en général par d'autres considérations géométriques (*Comptes Rendus*, 1883, Vol. xcvi., p. 1137)]..... 77

14921. (Rev. T. Wiggins, B.A.)—At right angles to a passage $5a$ feet wide is another passage $a\sqrt{3}$ feet wide. Find the greatest length of a rectangular table a feet wide which can be wheeled from one passage into the other 70

14922. (Robert W. D. Christie.)—Find a multiplier M which will make the modular equation $x^y - 1 = M(1/x^y - 1)$ true for any prime whatever, *e.g.*, $y = 6$, $x = 7$, $M = 8$, $7^6 - 1 = 8\,(\frac{1}{7} - 1)$, *i.e.*, $9 - 1 = \frac{8}{7} - 8$ or $-9 = 8 \bmod 17$ 115

14925. (Salutation.)—Through a particular vertex of an irregular (but convex) n-gon draw a straight line bisecting the figure..... 50

14927. (A. M. Nesbitt, M.A.)—Defining a cardioide as the locus of the extremities of a straight rod of length $2a$, which moves so that its middle point describes a circle of diameter a, while the rod is constrained to pass through a point on the circumference of the circle, prove *geometrically* that, "If three circles through the cusp of a cardioide touch the curve, their three points of intersection are collinear" ... 25

14932. (Professor Cochez.)—On donne un cercle $x^2 + y^2 = R^2$, et un point fixe A, par lequel on mène une sécante ABC. On construit le rayon OB et la perpendiculaire OM à OB rencontrante la sécante en M. Lieu de M 38

14937. (Professor S. Sircom.)—D, E, F are any points taken along the sides BC, CA, AB respectively, of the triangle ABC. Prove that (AE.BF.CD)/(AF.BD.CE) is unaltered by projection..... 27

14941. (R. Knowles.)—Tangents are drawn to an ellipse, whose major axis is AA′, from any point in the ordinate from the focus nearest A; tangents from A, A′ meet these in MN, M′N′, respectively. Prove that MN : M′N′ = $1 - e$: $1 + e$, e being the eccentricity of the ellipse 37

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14945. (W. C. Stanham.)—Solve in positive integers $x^3 - y^3 = 17z^3$ 38

14947. (R. Tucker, M.A.) —PQ, a chord of a parabola, passes through a fixed point O, on the axis : the normals at P, Q intersect in R. Show that the locus of R, for varying positions of PQ, is a parabola, and also that the envelope of these parabolas, for varying positions of O, is the evolute of the primitive parabola 94

14959. (R. Knowles, B.A.)—P, Q are the ends of a diameter of a conic, centre C ; R, R′ are the points through which pass all the chords which subtend a right angle at P, Q respectively. Prove that R, R′, C are collinear 64

14962. (Rev. T. Roach, M.A.)—Tom, Dick, and Harry set out on a journey into the wilderness carrying a small flag which is to be carried forward as many days' journey as possible. Tom and Dick carry 60 rations each and Harry carries 30. Tom and Dick eat 3 rations a day each and Harry eats $1\frac{1}{2}$. They can put down any part of their loads, to be taken up on their return, but cannot return for a second load. Find the farthest point to which they can get the flag..... 84

14966. (Rev. Prebendary W. Allen Whitworth.)—If a point P be taken at random within a trapezium ABCD, the expectation of the area PAB is

$$\tfrac{1}{3}\mathrm{ABC}\,.\,\mathrm{ABD}\left\{1/\mathrm{ABC}+1/\mathrm{ABD}-1/\mathrm{ABCD}\right\} \text{ ..... } 72$$

14967. (R. Tucker, M.A.)—The tangent at P (ap^2, ap^3) to the curve $ay^2 = x^3$ (A) meets the curve again at Q ; prove that (1) the normals at P, Q meet, in R, on the curve $6561y^4 + 2430axy^2 + 32a^2y^2 = ax\,(75x + 8a)^2$; (2) the evolute to (A) is $9y^2\,(729y^2 + 1728ax + 32a^2) + 512ax\,(6x + a)^2 = 0$. (Gregory's *Exercises*, p. 195, gives a surd result.) If $9p^2 = 2$, then PR is also a tangent. Find the locus of the intersection of orthogonal tangents, and also of the circum-centre of POQ (O the cusp), and show that the mid-point of PQ lies on another semicubical parabola 66

14968. (Major P. A. MacMahon, F.R.S.)—Prove that that portion of the expansion of the algebraic fraction

$$\frac{1}{\left\{1-s_1(a_1x_1+a_2x_2)\right\}^p\left\{1-s_2(b_1x_1+b_2x_2)\right\}^q} \quad (p \geqslant q)$$

which is a function of s_1x_1 and s_2x_2 only has the expressions

$$\frac{1}{\mathrm{D}^q\,(1-s_1a_1x_1)^{p-q}}\left\{1+\binom{p-1}{1}\binom{q}{1}\frac{d}{\mathrm{D}}+\binom{p-1}{2}\binom{q+1}{2}\frac{d^2}{\mathrm{D}^2}\right.$$
$$\left.+\binom{p-1}{3}\binom{q+2}{3}\frac{d^3}{\mathrm{D}^3}+\dots\right\},$$

where $\mathrm{D} = 1-a_1s_1x_1-b_2s_2x_2+(a_1b_2-a_2b_1)\,s_1s_2x_1x_2$ and $d = a_2b_1s_1s_2x_1x_2$ 32

14970. (Lt.-Col. Allan Cunningham, R.E.)—Show how to find a set of three or more numbers q_r such that $(x^{q_0}-1)$, $(x^{q_1}-1)$, $(x^{q_2}-1)$, &c., are divisible by q_1, q_2, q_3, &c., *respectively* for several assigned values of x (= a, b, c, &c.) *at once*. Give examples when x has *each* of the values 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30 for the same set of exponents q_0, q_1, q_2, &c., with the respective divisors q_1, q_2, q_3, &c. 64

14971. (Robert W. D. Christie.)—Find factors of N^4+1 by an easy general solution where N^2 takes the form $(\mathrm{M}^2-24)/10$. (See Quest. 13707.) Ex. $14506^4 + 1 = 210469913$ and 210378169 ..... 33, 53

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14974. (R. Chartres.) — In a triangle ABC, if $\cos A = \frac{1}{3}$, and a straight line be drawn from A making an angle $\frac{1}{4}\pi$ with its bisector, then the parts intercepted without and within the in-circle are equal to its radius, and the parts intercepted without and within the ex-circle are equal to its radius 87

14978. (Rev. T. Roach, M.A.) — Circles are described on SP, S'P in an ellipse as diameters and meet the auxiliary circle in R, R'; show that RR' is the tangent at P 62

14980. (R. Knowles, B.A.)—Prove that the sum to r terms of the series $1 + 2^3x + 3^3x^2 + \ldots + r^3x^{r-1}$ is

$$\{1 + 4x + x^2 - (r+1)^3x^r + (3r^3 + 6r^2 - 4)\,x^{r+1} - (3r^3 + 3r^2 - 3r + 1)\,x^{r+2} + r^3x^{r+3}\}/(1-x)^4$$

..... 54

14985. (Professor Neuberg.) — Ramener l'un à l'autre les déterminants

$$\begin{vmatrix} x_1+x_2-x_3-x_4 & y_1+y_2-y_3-y_4 & x_1x_2+y_1y_2-x_3x_4-y_3y_4 \\ x_1+x_3-x_2-x_4 & y_1+y_3-y_2-y_4 & x_1x_3+y_1y_3-x_2x_4-y_2y_4 \\ x_1+x_4-x_2-x_3 & y_1+y_4-y_2-y_3 & x_1x_4+y_1y_4-x_2x_3-y_2y_3 \end{vmatrix},$$

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix}$$

..... 32

14988. (The late Professor Cayley, F.R.S.) — If $l^2+m^2+n^2+\ldots$, $l'^2+m'^2+n'^2+\ldots$ are each = 0, and s, s' denote positive integers, show that the integral

$$\int (lx+my+nz+\ldots)^s\,(l'x+m'y+n'z+\ldots)^{s'}\,dx\,dy\,dz\ldots$$

over the interior of the hyper-sphere $x^2+y^2+z^2+\ldots=1$ is = 0 or = a numerical multiple of $(ll'+mm'+nn'+\ldots)^s$, according as s' is not, or is, $= s$ 59

14991. (Professor Umes Chandra Ghosh.) — If $f(r_1r_2) = 0$ be the equation of a plane curve, where r_1 and r_2 are the distances of any point on the curve from two fixed points S_1, S_2 in its plane, and if ρ_1, ρ_2 be the distances of any point on a normal to the curve from S_1, S_2, show that

$$\left[\frac{\rho_1^2-\rho_2^2-(a_1^2-a_2^2)}{r_1^2-r_2^2-(a_1^2-a_2^2)}\right]^2 = \frac{a\,(\rho_1^2-a_1^2)+a_1[(a_1^2-a_2^2)-(\rho_1^2-\rho_2^2)]}{a\,(r_1^2-a_1^2)+a_1[(a_1^2-a_2^2)-(r_1^2-r_2^2)]},$$

where $a_1 = a\,\dfrac{r_1\,(\partial f/\partial r_2)}{r_1\,(\partial f/\partial r_2)+r_2\,(\partial f/\partial r_1)}$, $a_2 = a\,\dfrac{r_2\,(\partial f/\partial r_1)}{r_1\,(\partial f/\partial r_2)+r_2\,(\partial f/\partial r_1)}$,

and $S_1S_2 = a$ 117

14993. (Lt.-Col. Allan Cunningham, R.E.)—Let $\sigma(N)$ denote the *sum of the divisors* of a number N (including 1, but not N itself). Given $\sigma(N)$, to find N. Is this *always possible?* When is the solution *unique*, and when are there *multiple* solutions? Give examples..... 98

15001. (A. M. Nesbitt, M.A.)—Two conics U and V intersect in ABCO, and through O two arbitrary straight lines OP, OQ are drawn; OP cuts U in P and V in Q'; OQ cuts U in P' and V in Q. Prove that the four straight lines OP, OQ, PP', QQ' and the three sides of the triangle ABC all touch one and the same conic 113

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15006. (Professor Ignacio Beyens, Lt.-Col. du Génie.)—Soit SABC un angle trièdre trirectangle ; SD une droite intérieure que l'on projette sur les faces SAB, SAC, SBC, suivant SC′, SB′, SA′ ; démontrer que A′SB′ + A′SC′ + B′SC′ = 180°, et déduire de ce théorème la propriété correspondante du triangle sphérique (démonstration géométrique)... 63

15014. (Professor Nanson.)—A conic touches BC, CA, AB in D, E, F. If DO, EO, FO cut the conic again in P, Q, R, prove AP, BQ, CR concurrent 71, 75

15015. (Rev. T. Roach, M.A.)—If PQ, perpendicular to SP in the ellipse, meets CD in Q and S′P meets CD in E, show that the normal at P will touch the circle about PEQ in P 75

15023. (R. Chartres.)—If C_r denote the combinations of n things r together, and D_r those of $(2n-1)$ things r together, sum the series

$$\frac{C_1C_1}{D_1} - \frac{C_2C_3}{D_3} + \frac{C_3C_5}{D_5} - \frac{C_4C_7}{D_7} + \dots \quad \dots\dots 60$$

15029 (Professor H. Langhorne Orchard, M.A., B.Sc.)—Show that the value of

$$\left[\int_0^\infty \left(\frac{x^4+9}{3x^2}\right) e^{-(x^4+9)/3x^2}\,dx\right] \Big/ \left[\int_0^\infty \left(\frac{x^4-9}{3x^2}\right) e^{-(x^4+9)\cdot 3x^2}\,dx\right]$$

lies between 4 and 6 104

15040. (R. Chartres.)—Find the mean value of the n-th power of the area of a cyclic quadrilateral whose perimeter $(2s)$ is constant, and deduce the results for (1) a kite, (2) a triangle..... 81

15042. (H. Bateman.)—Eliminate A, B, C from the equations

$$\sin 2A \sin A = l \sin B \sin C, \qquad \sin 2B \sin B = m \sin C \sin A,$$
$$\sin 2C \sin C = n \sin A \sin B. \qquad A + B + C = \pi \quad \dots\dots 77$$

15045. (E. V. Huntington, Ph.D.)—The centre of gravity of the equally weighted vertices and ortho-centre of any triangle is the centre of the nine-point circle of that triangle 93

15061. (J. Prescott.)—From the fact that the inverse of a plane is a sphere through the centre of inversion, deduce that the inverse of a circle, with respect to a point not in its plane, is a circle 86, 95

15064. (H. MacColl, B.A.)—Let ϕ denote the alternative statement $xyza' + xyb' + xy'z' + y'z'a'$. Express the implication $\epsilon : \phi$ (which asserts that ϕ is *certain*, *i.e.*, follows necessarily from our data) in the form $(M : z : N)(P : y : Q)(R : z : S)(T : \eta)$; in which M denotes the weakest statement from which (in conjunction with our data) we can deduce z, and N denotes the strongest conclusion we can draw from z. Similarly P, Q, R, S are to be interpreted. It is understood that M and N may contain y and x, but not z; that P and Q may contain x, but neither y nor z; and that R, S, T contain neither x nor y nor z, though they may contain a or b. The implication $T : \eta$ asserts that T is *impossible*. *i.e.*, inconsistent with our data and definitions. The symbol A′ is the denial of A, so that, whatever be the statement A, we have always $\epsilon : A + A'$ and $AA' : \eta$. Logicians will notice the connexion between this Question and the "inverse problem" of Prof. Jevons; and mathematicians will notice the analogy between it and the problem to find the successive limits of integration in multiple integrals 99

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15070. (C. F. Sandberg, M.A.)—If, for all values of r from 0 to n,

$$a_r = a_r + ra_{r-1}x + \frac{r(r-1)}{1.2}a_{r-2}x^2 + \ldots + ra_1x^{r-1} + a_0x^r,$$

then, for the same values of r,

$$a^r = a_r - ra_{r-1}x + \frac{r(r-1)}{1.2}a_{r-1}x^2 + \ldots + (-1)^r a_0x^r \ldots\ldots 103$$

15078. (Professor Hudson, M.A.)—If a uniform chain of given length be fastened to two points at a given distance apart in the same horizontal line, and be acted on by vertical forces such that the force acting on a length ds is $mgds + \mu gdx$, prove that the horizontal tension is approximately $(m+\mu)\,g\,\{x^3/6\,(s-x)\}^{\frac{1}{2}}$, and that the sag at the middle point is $\{\frac{3}{2}x(s-x)\}^{\frac{1}{2}}$, assuming that $(s-x)/x$ is small 90

15089. (The late Professor R. B. Clayton.) — Normals lying in a given plane are drawn to a series of confocal quadrics. Show that they envelop a parabola..... 114

15099. (R. Knowles.)—From a point T (h, k) tangents are drawn to the parabola $y^2 = 4ax$; from a point P (x', y') a third tangent is drawn meeting these in M and N. Prove that the equation to the circle TMN is $x^2 + y^2 - \frac{ky' + 2a\,(a+h)}{2a}\,x - \frac{2ak + y'\,(a-h)}{2a}\,y + \frac{ky' + 2ah}{2} = 0$ ... 117

15101. (Professor Cochez.) — On donne une circonférence C, $x^2 + y^2 = R^2$, et la droite Δ $(x = a)$. On construit la circonférence C′ tangente à C et à Δ. La tangente commune T rencontre le diamètre C′A en M. Lieu de M 116

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MATHEMATICS

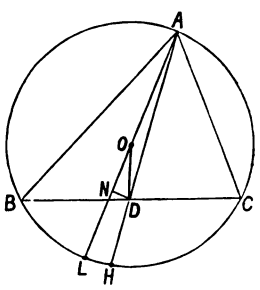
FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

Geometrical Note by R. F. Davis, M.A.

Let ABC be a triangle, O its circum-centre, and D the middle point of its base BC. Produce AO, AD to meet the circum-circle again in L, H; draw DN perpendicular to AL. Then



$$AB^2 + AC^2$$
$$= AL \text{ (sum of projections of AB, AC upon AL)}$$
$$= AL \text{ (twice projection of AD)}$$
$$= 2AL \cdot AN$$
$$= 2AH \cdot AD \quad \text{(since DHLN is cyclic)}$$
$$= 2(AD^2 + BD^2).$$

If D, instead of being the middle point of BC, divides it so that

$$BD : CD = q : p,$$

then $p \cdot AB^2 + q \cdot AC^2 = (p+q)\, AL \cdot AN = (p+q)(AD^2 + BD \cdot CD)$, as before.

14927. (A. M. Nesbitt, M.A.)—Defining a cardioide as the locus of the extremities of a straight rod of length $2a$, which moves so that its middle point describes a circle of diameter a, while the rod is constrained to pass through a point on the circumference of the circle, prove *geometrically* that, "If three circles through the cusp of a cardioide touch the curve, their three points of intersection are collinear."

Solutions (I.) *by the* Proposer; (II.) *by* John Prescott.

(I.) If OQC_1R be generating circle, centre E, and OR be perpendicular to the rod PP′, then RP is normal to cardioide at P, and Q is mid-point

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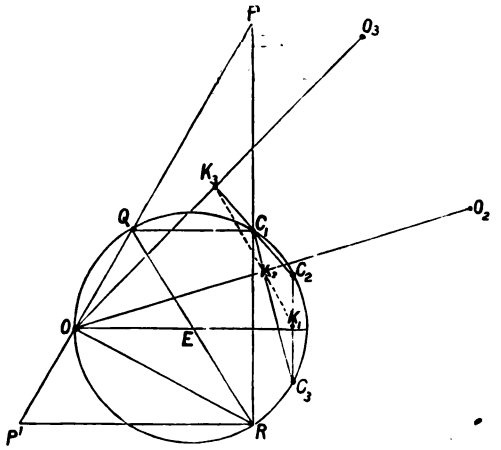
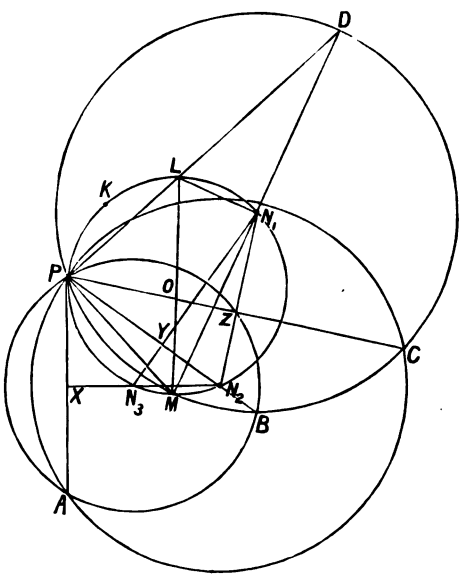
of rod, so that C_1 is centre of circle through O which touches cardioide



at P. Thus the centres of the three circles are (C_1, C_2, C_3) three points on generating circle; and common chord of circles with centres C_1, C_2 will be OO_3, where O_3 is image of O in C_1C_2, *i.e.* $OO_3 = 2OK_3$. But $K_1K_2K_3$ is the SIMSON line of O with respect to triangle $C_1C_2C_3$; therefore O_1, O_2, O_3 are collinear points.

(II.) Let PKL be the fixed circle and P the fixed point on it. Join PL



and produce it to D, so that LD = a. Draw the diameter LOM and join

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MP; then M is known to be the instantaneous centre of rotation of the rod and D is a point on the cardioide.

Now it follows that the direction of motion of D, *i.e.*, the tangent to the cardioide, is perpendicular to MD. Hence the centre of a circle touching the cardioide at D lies in MD. Let N_1 be the point where MD meets the circle PKL. Join PN_1. Then N_1P, N_1D, N_1M are all equal. For LN_1 is perpendicular to MN_1, and LM = LD; therefore N_1 is the middle point of MD. Also DPM is a right angle; hence the circle described on MD as diameter passes through P, and this circle touches the cardioide and passes through its cusp P. Hence every circle which touches the cardioide and passes through the cusp has its centre on the circumference of the circle PKL.

Let N_2, N_3 be two other points on the circumference of the circle PKL, and let circles be described with N_1, N_2, N_3 as centres and N_1P, N_2P, and N_3P as radii; and let the intersections of circles whose centres are N_2 and N_3, N_1 and N_3, N_1 and N_2 be A, B, C respectively.

It is required to prove that A, B, and C are collinear.

Join PA, PB, PC, N_2N_3, N_1N_3, N_1N_2.

Now, since N_2 and N_3 are the centres of two circles, the line N_2N_3 bisects at right angles PA, their common chord.

Similarly, N_1N_3 and N_1N_2 bisect at right angles PB and PC respectively.

Let X, Y, and Z be the middle points of PA, PB, and PC respectively.

Now PX, PY, and PZ are perpendiculars to the sides of a triangle from a point on the circumscribing circle; therefore X, Y, and Z are collinear: and, since BA is parallel to XY, and BC is parallel to YZ, it follows that A, B, and C are collinear.

14937. (Professor S. Sircom.)—D, E, F are any points taken along the sides BC, CA, AB respectively, of the triangle ABC. Prove that (AE.BF.CD)/(AF.BD.CE) is unaltered by projection.

Solution by H. C. Plummer.

Let AD, BE meet in O; CO and AB in F'. Let

$$(AE.BF.CD)/(AF.BD.CE) = x.$$

But $$(AE.BF'.CD)/(AF'.BD.CE) = 1.$$

Therefore $$x = (BF.AF')/(AF.BF') = \{ABFF'\},$$

and is unaltered by projection.

4289 & 6155. (Professor Cayley, F.R.S.)—Given, by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance.

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Solution by Professor NANSON.

The metrical coordinates of the line $\xi-x : \eta-y : \zeta-z = a : b : c$ are a, b, c, f, g, h, where $f = yc-zb$, ..., and $a^2+b^2+c^2 = 1$, $af+bg+ch = 0$. Hence, if θ be the angle and δ the shortest distance between two lines,

$$\cos\theta = aa'+bb'+cc', \quad \sin^2\theta = (bc')^2+(ca')^2+(ab')^2,$$

$$\delta = (x-x', a, a')\operatorname{cosec}\theta = [(x, a, a')+(x', a', a)]\operatorname{cosec}\theta$$

$$= (af'+a'f+\ldots)\operatorname{cosec}\theta.$$

Also, if A, F are the coordinates of the shortest distance,

$$Aa+\ldots = 0, \qquad Aa'+\ldots = 0,$$

$$fA+aF+\ldots = 0, \quad f'A+a'F+\ldots = 0,$$

$$A^2+\ldots = 1, \qquad AF+\ldots = 0,$$

whence $A = (bc')\operatorname{cosec}\theta$, ..., and, if $\Delta = (a, a', f)$, $\Delta' = (a', a, f')$, then

$$aF+\ldots = -\Delta\operatorname{cosec}\theta, \quad a'F+\ldots = \Delta'\operatorname{cosec}\theta;$$

therefore $$(a\Delta'+a'\Delta)F+\ldots = 0;$$

therefore $$F = \mu[\Delta(b'C-c'B)+\Delta'(bC-cB)], \ \ldots,$$

$$= \lambda[\Delta(a-a'\cos\theta)-\Delta'(a'-a\cos\theta)], \ \ldots,$$

and $$\lambda = -\operatorname{cosec}^2\theta.$$

14693. (J. J. BARNIVILLE, B.A.)—Having $u_n = u_{n-2}+u_{n-3}$, prove that

$$\frac{1}{2.4.5}+\frac{1}{3.5.7}+\frac{1}{4.7.9}+\frac{1}{5.9.12}+\ldots = \frac{1}{24},$$

$$\frac{1}{2.3.4.5.9}+\frac{1}{3.4.5.7.12}+\frac{1}{4.5.7.9.16}+\ldots = \frac{1}{840},$$

$$\frac{2.4.12}{3.5.7.9}+\frac{3.5.16}{4.7.9.12}+\frac{4.7.21}{5.9.12.16}+\ldots = \frac{1}{3}+\frac{1}{4}-\frac{1}{7},$$

$$\frac{2^2}{1.3.5}+\frac{3^2}{1.4.7}+\frac{4^2}{1.5.9}+\frac{5^2}{2.7.12}+\ldots = \frac{41}{24},$$

$$\frac{1}{9^2-1^2}+\frac{2}{12^2-2^2}+\frac{2}{16^2-2^2}+\frac{3}{21^2-3^2}+\ldots = \frac{1}{16}.$$

Solution by Professor SANJÁNA, M.A.

We have

$$\text{(i.)} = \frac{3}{2.3.4.5}+\frac{4}{3.4.5.7}+\frac{5}{4.5.7.9}+\ldots$$

$$= \frac{5-2}{2.3.4.5}+\frac{7-3}{3.4.5.7}+\frac{9-4}{4.5.7.9}+\ldots \quad [\text{since } u_{n-2} = u_n-u_{n-1}]$$

$$= \frac{1}{2.3.4}-\frac{1}{3.4.5}+\frac{1}{3.4.5}-\frac{1}{4.5.7}+\frac{1}{4.5.7}-\ldots = \frac{1}{24}.$$

As $$u_n = u_{n-2}+u_{n-3} = u_{n-4}+u_{n-5}+u_{n-3} = u_{n-1}+u_{n-5},$$

therefore $$u_{n-1} = u_n-u_{n-5}.$$

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Hence (ii.) $= \frac{7}{2.3.4.5.7.9} + \frac{9}{3.4.5.7.9.12} + \frac{12}{4.5.7.9.12.16} + \dots$

$= \frac{9-2}{2.3.4.5.7.9} + \frac{12-3}{3.4.5.7.9.12} + \frac{16-4}{4.5.7.9.12.16} + \dots$

$= \frac{1}{2.3.4.5.7} = \frac{1}{840}.$

Again, $\frac{1}{3} - \frac{1}{5} = \frac{2}{3.5}, \quad \frac{1}{7} - \frac{1}{9} = \frac{2}{7.9};$

therefore $\frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = \frac{2.48}{3.5.7.9} = \frac{2.4.12}{3.5.7.9}.$

So $\frac{1}{4} - \frac{1}{7} - \frac{1}{9} + \frac{1}{12} = \frac{3.5.16}{4.7.9.12}, \quad \frac{1}{5} - \frac{1}{9} - \frac{1}{12} + \frac{1}{16} = \frac{4.7.21}{5.9.12.16}, \dots.$

Thus (iii.) $= \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{4} - \frac{1}{7} - \frac{1}{9} + \frac{1}{12} + \frac{1}{5} - \frac{1}{9} - \frac{1}{12} + \frac{1}{16}$

$+ \frac{1}{7} - \frac{1}{12} - \frac{1}{16} + \frac{1}{21} + \dots = \frac{1}{3} - \frac{1}{7} + \frac{1}{4}.$

Also (iv.) $= \frac{2}{5}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{3}{7}\left(\frac{1}{1} - \frac{1}{4}\right) + \frac{4}{9}\left(\frac{1}{1} - \frac{1}{5}\right) + \frac{5}{12}\left(\frac{1}{2} - \frac{1}{7}\right) + \dots$

$= \frac{2}{1.5} + \frac{3}{1.7} + \frac{4}{1.9} + \frac{5}{2.12} + \frac{7}{2.16} + \dots$

$- \left(\frac{2}{3.5} + \frac{3}{4.7} + \frac{4}{5.9} + \frac{5}{7.12} + \dots\right)$

$= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{5} + \frac{1}{1} - \frac{1}{7} + \frac{1}{1} - \frac{1}{9} + \frac{1}{2} - \frac{1}{12} + \frac{1}{2} - \frac{1}{16} + \dots\right)$

$- \left(\frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{7} + \frac{1}{5} - \frac{1}{9} + \dots\right)$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right)$

$= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) = \frac{41}{24}.$

Similarly,

(v.) $= \frac{1}{2}\left(\frac{1}{8} - \frac{1}{10}\right) + \frac{1}{2}\left(\frac{1}{10} - \frac{1}{14}\right) + \frac{1}{2}\left(\frac{1}{14} - \frac{1}{18}\right) + \dots = \frac{1}{16}.$

14751. (H. MacColl, B.A.)—The chance that A is true (assuming nothing as to B) is a; the chance that B is true (assuming nothing as to A) is b; and the chance that A is true, assuming B true, is k times the chance that B is false, assuming A false. Show that the dependence of A upon B is $(1-a)(k-kb-a)/(1-a-kb)$, except when the numerator and denominator of this fraction both vanish. Discuss this exceptional case. (See *Reprint*, Vol. LXXII., pp. 79, 80.)

Solution by the Proposer.

We assume that A and B are neither certainties nor impossibilities; so

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that a and b must each be >0 and <1. Let $a' = 1-a$ and $b' = 1-b$. From the formulæ $A'/B = 1-A/B$, and $A/B = a/b \,.\, B/A$, we get

$$\frac{B'}{A'} = 1-\frac{B}{A'} = 1-\frac{b}{a'}\cdot\frac{A'}{B} = 1-\frac{b}{a'}\left(1-\frac{A}{B}\right) = 1-\frac{b}{a'}+\frac{b}{a'}\cdot\frac{A}{B}.$$

And from our data we have $A/B = k\,B'/A'$. These two equations give us

$$A/B = k\,(a'-b)/(a'-kb);$$

and, by definition, $\delta\,(A/B) = A/B-a = (1-a)(k-kb-a)/(1-a-kb)$.

The numerator and denominator of this fraction both vanish when $k = a+b = 1$. In this case $\delta\,(A/B)$ may have an infinite number of values all consistent with the data. For example, let $a = \frac{1}{3}$, $b = \frac{1}{3}$, $k = 1$. To illustrate geometrically, let x points (see Fig. 1) be marked in the circle A outside the circle B, y points in B outside A, 6 points in the area common to A and B, and 6 points outside both A and B. Let the symbol A as a statement assert that a point P taken at random out of the whole $x+y+12$ points marked in the circle E will be one of the $x+6$ in A, and let B assert that it will be one of the $y+6$ in B. From our data

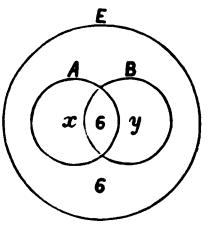

Fig. 1.

$$(x+6)/(x+y+12) = a = \tfrac{1}{3},$$
$$(y+6)/(x+y+12) = b = \tfrac{1}{3}.$$

These two equations are not independent, each being equivalent to $2x-3y = 6$, so that an infinite number of values of x and y will satisfy them; and, whatever values of these we take, the expression given for $\delta\,(A/B)$ will assume the form $\frac{0}{0}$. Yet all the values of $\delta\,(A/B)$ corresponding to the arbitrary values assigned to x and y will be real and different. For example, when $x = 6$ and $y = 2$, we get $\delta\,(A/B) = \frac{3}{10}$. When $x = 9$ and $y = 4$, $\delta\,(A/B) = 0$. When $x = 12$ and $y = 6$, $\delta\,(A/B) = -\frac{1}{10}$. And for all these different values of $\delta\,(A/B)$ we have the same data, $a = \frac{1}{3}$, $b = \frac{1}{3}$, $k = 1$; which shows that these data alone are insufficient to determine $\delta\,(A/B)$.

But when the number of points marked in the area common to A and B is not equal to the number outside both A and B the formula for $\delta\,(A/B)$ will *not* take the form $\frac{0}{0}$, and will always determine the one and only possible value of the dependence $\delta\,(A/B)$. For example, suppose (as in Fig. 2) we have $a = \frac{1}{3}$, $b = \frac{1}{4}$, $k = \frac{1}{3}$. It is evident from the figure that $\delta\,(A/B) = -\frac{1}{5}$, and this result will also be obtained from the formula without any reference to the figure. Any values of a, b, k which will make $\delta\,(A/B)$ in the formula greater than 1 will indicate inconsistency of data, as, for example, $a = \frac{1}{3}$, $b = \frac{1}{4}$, $k = \frac{2}{3}$. When $a = \frac{1}{3}$ and $b = \frac{1}{4}$, k cannot exceed $\frac{10}{9}$.

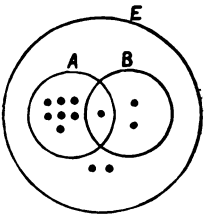

Fig. 2.

14655. (J. A. Third, M.A., D.Sc.)—From the vertices of a triangle ABC lines AX, BY, CZ are drawn to the opposite sides, such that the

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angles CAX, ABY, BCZ, taken the same way round, are together equal to 180°; and AX′, BY′, CZ′ are drawn from the vertices to the opposite sides, parallel to CZ, AX, BY respectively. Show that the circum-centres of AXX′, BYY′, CZZ′ are collinear.

Solution by Professor SANJÂNA, M.A.

Suppose through A two lines are drawn to the base BC, such that $CAX' = \phi_1$, $BAX = \theta_1$; let us find the trilinear coordinates of the circum-centre of AXX′. The coordinates of A are $(p_1, 0, 0)$; of X, $[0, p_1 \sin(A-\theta_1)/\sin(B+\theta_1), p_1 \sin\theta_1/\sin(B+\theta_1)]$; hence those of the mid-point of AX are

$$[1, \sin(A-\theta_1)/\sin(B+\theta_1), \sin\theta_1/\sin(B+\theta_1)],$$

omitting the factor $\frac{1}{2}p_1$. The line through A perpendicular to AX is

$$\beta\cos\theta_1 + \gamma\cos(A-\theta_1) = 0;$$

hence the perpendicular bisector of AX is

$$\beta\cos\theta_1 + \gamma\cos(A-\theta_1) = k(\alpha\sin A + \beta\sin B + \gamma\sin C),$$

where k may be determined from the fact that the mid-point above lies on this line; this gives for k the value $1/2\sin(B+\theta_1)$, and we get for this bisector the equation

$$\beta\sin(B+2\theta_1) + \gamma\sin(B-A+2\theta_1) - \alpha\sin A = 0.$$

Similarly the perpendicular bisector of AX′ is found to be

$$\gamma\sin(C+2\phi_1) + \beta\sin(C-A+2\phi_1) - \alpha\sin A = 0.$$

Solving these equations simultaneously, we obtain for O_1, the circum-centre of AXX′,

$$\alpha/\cos(A-\theta_1-\phi_1) = \beta/\cos(B-\phi_1+\theta_1) = \gamma/\cos(C-\theta_1+\phi_1) \quad \ldots (1).$$

So, also, if lines through B are drawn to CA such that $ABY' = \phi_2$, $CBY = \theta_2$, and through C to AB such that $BCZ' = \phi_3$, $ACZ = \theta_3$, the coordinates of O_2 and O_3, the circum-centres of BYY′ and CZZ′ respectively, are

$$\alpha/\cos(A+\phi_2-\theta_2) = \beta/\cos(B-\theta_2-\phi_2) = \gamma/\cos(C+\theta_2-\phi_2) \quad \ldots (2),$$

$$\alpha/\cos(A+\theta_3-\phi_3) = \beta/\cos(B+\phi_3-\theta_3) = \gamma/\cos(C-\theta_3-\phi_3) \quad \ldots (3).$$

In the present case, as $CAX + ABY + BCZ = 180°$, we get $ACZ = BAX + CBY$, that is, $-\theta_3 = \theta_1 + \theta_2$; also, as BY′ is parallel to AX, $\phi_2 = -\theta_1$; so $\theta_3 = -\phi_1$ and $\phi_3 = -\theta_2 = \theta_1 + \theta_3 = \theta_1 - \phi_1$. Thus the three circum-centres are given by (1) and

$$\alpha/\cos(A-\phi_1) = \beta/\cos(B+2\theta_1-\phi_1) = \gamma/\cos(C+\phi_1) \ldots\ldots (2')$$

and

$$\alpha/\cos(A-\theta_1) = \beta/\cos(B+\theta_1) = \gamma/\cos(C+2\phi_1-\theta_1) \ldots\ldots (3').$$

The line joining the last two will be found to be

$$\alpha\sin(B-C+2\theta_1-2\phi_1) + \beta\sin(C-A+2\phi_1) + \gamma\sin(A-B-2\theta_1) = 0,$$

and O_1 may be easily shown to lie upon this.

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14985. (Professor NEUBERG.)—Ramener l'un à l'autre les déterminants

$$\begin{vmatrix} x_1+x_2-x_3-x_4 & y_1+y_2-y_3-y_4 & x_1x_2+y_1y_2-x_3x_4-y_3y_4 \\ x_1+x_3-x_2-x_4 & y_1+y_3-y_2-y_4 & x_1x_3+y_1y_3-x_2x_4-y_2y_4 \\ x_1+x_4-x_2-x_3 & y_1+y_4-y_2-y_3 & x_1x_4+y_1y_4-x_2x_3-y_2y_3 \end{vmatrix},$$

$$\begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix}.$$

Solution by Dr. MUIR, F.R.S.

Calling the first determinant D_3 and the second D_4 and denoting $x_rx_s+y_ry_s$ by σ_{rs}, we increase the fourth column of D_4 by $\sigma_{23}+\sigma_{24}+\sigma_{34}$ times the first column, and diminish it by Σx times the second column, and by Σy times the third column, the result being

$$\begin{vmatrix} 1 & x_1 & y_1 & -\sigma_{12}-\sigma_{13}-\sigma_{14}+\sigma_{23}+\sigma_{24}+\sigma_{34} \\ 1 & x_2 & y_2 & -\sigma_{12}+\sigma_{34} \\ 1 & x_3 & y_3 & -\sigma_{13}+\sigma_{24} \\ 1 & x_4 & y_4 & -\sigma_{14}+\sigma_{23} \end{vmatrix}.$$

If this be multiplied columnwise by -8 in the form

$$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix},$$

we obtain

$$-8D_4 = \begin{vmatrix} 2 & -x_1+x_2+x_3+x_4 & -y_1+y_2+y_3+y_4 & 0 \\ 0 & x_1+x_2-x_3-x_4 & y_1+y_2-y_3-y_4 & 2(-\sigma_{12}+\sigma_{34}) \\ 0 & x_1-x_2+x_3-x_4 & y_1-y_2+y_3-y_4 & 2(-\sigma_{13}+\sigma_{24}) \\ 0 & x_1-x_2-x_3+x_4 & y_1-y_2-y_3+y_4 & 2(-\sigma_{14}+\sigma_{23}) \end{vmatrix} = -4D_3;$$

and therefore

$$2D_4 = D_3.$$

14968. (Major P. A. MACMAHON, F.R.S.)—Prove that that portion of the expansion of the algebraic fraction

$$\frac{1}{\{1-s_1(a_1x_1+a_2x_2)\}^p\{1-s_2(b_1x_1+b_2x_2)\}^q} \quad (p \geqslant q)$$

which is a function of s_1x_1 and s_2x_2 only has the expressions

$$\frac{1}{D^q(1-s_1a_1x_1)^{p-q}}\left\{1+\binom{p-1}{1}\binom{q}{1}\frac{d}{D}+\binom{p-1}{2}\binom{q+1}{2}\frac{d^2}{D^2}\right.$$
$$\left.+\binom{p-1}{3}\binom{q+2}{3}\frac{d^3}{D^3}+\dots\right\},$$

where $D = 1-a_1s_1x_1-b_2s_2x_2+(a_1b_2-a_2b_1)s_1s_2x_1x_2$ and $d = a_2b_1s_1s_2x_1x_2$.

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Solution by Professor E. B. Elliott, F.R.S.

Writing a_1, a_2, β_1, β_2 for $s_1a_1x_1$, $s_1a_2x_1$, $s_2b_1x_2$, $s_2b_2x_2$, respectively, and y for x_2/x_1, the problem is to extract from

$$\{1-(a_1+a_2y)\}^{-p}\{1-(\beta_2+\beta_1y^{-1})\}^{-q}$$

the part whose expansion in powers of a_1, a_2, β_1, β_2 is free from y. The part is clearly the infinite expansion

$$(1-a_1)^{-p}(1-\beta_2)^{-q}\left\{1+pq\frac{a_2\beta_1}{(1-a_1)(1-\beta_2)}+\frac{p(p+1)}{1.2}\frac{q(q+1)}{1.2}\frac{a_2^2\beta_1^2}{(1-a_1)^2(1-\beta_2)^2}+\ldots\right\};$$

but a finite expression for it is desired.

Now the expansion here is also the part free from y in that of

$$(1-a_1)^{-p}(1-\beta_2)^{-q}(1-y)^{-p}\left\{1-\frac{a_2\beta_1}{(1-a_1)(1-\beta_2)}y^{-1}\right\}^{-q},$$

i.e., of $$(1-a_1)^{q-p}(1-y)^{-p}\{(1-a_1)(1-\beta_2)-a_2\beta_1y^{-1}\}^{-q},$$

i.e., of $$(1-a_1)^{q-p}(1-y)^{-p}\{D+d(1-y^{-1})\}^{-q},$$

i.e., of $$D^{-q}(1-a_1)^{q-p}\left\{(1-y)^{-p}+q\frac{d}{D}(1-y)^{-p}(y^{-1}-1)+\frac{q(q+1)}{1.2}\frac{d^2}{D^2}(1-y)^{-p}(y^{-1}-1)^2+\ldots\right\}.$$

But, r being any positive integer, the part free from y in the expansion of $(1-y)^{-p}(y^{-1}-1)^r$ is that in the expansion of $(1/y^r)(1-y)^{r-p}$ which is zero unless $r<p$, and if $r<p$ is

$$\frac{(p-r)(p-r-1)\ldots(p-1)}{r!},\ \textit{i.e.},\ \binom{p-1}{r}.$$

Hence the finite expression required is as stated.

14971. (Robert W. D. Christie.)—Find factors of N^4+1 by an easy general solution where N^2 takes the form $(M^2-24)/10$. (See Quest. 13707.) Ex. $14506^4+1 = 210469913$ and 210378169.

Solution by Lt.-Col. Allan Cunningham, R.E.

The present Solver has given in a communication to the London Mathematical Society (as yet unpublished) a *method* of considerable generality of factorizing the functions $N=(x^4+y^4)$, called *Quartans;* the following is a slight sketch of what is *here* wanted.

Let $$N = x^4+y^4 = (x^2+Cy^2)^2-y^2\,.\,z^2 = P^2-Q^2.$$

This requires $$z^2 = 2Cx^2+(C^2-1)\,y^2.$$

If this diophantine be solvable in integers (x, y, z), it will yield an infinite train of factorisable quartans. Also, it was shown that every quartan, even if *prime*, would yield (several) values of C, each leading to a *solvable* diophantine.

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Ex.—Take $N_0 = 17 = 2^4 + 1^4 = 9^2 - 8^2$ as base-quartan. Here $x_0 = 2$, $y_0 = 1$, giving $2^2 + C \cdot 1^2 = \pm 9$, whence $C = 5$ or -9. Taking $C = 5$ gives the diophantine $z^2 - 10x^2 = 24y^2$. Assuming $y = 1$, the diophantine $z^2 - 10x^2 = 24$ will yield a train of $N = x^4 + 1^4$.

The initial solution $8^2 - 10 \cdot 2^2 = 24$, together with the unit-form $19^2 - 10 \cdot 6^2 = 1$, yields, by the usual rules, *two* infinite trains of solutions

$z = 8, 32, 1208, 45872$, &c., &c.; $272, 10328$, &c.;

$x = 2, 10, 382, 14506$, &c., &c.; $86, 3266$, &c.;

each solution giving the factorization of a quartan.

Thus the fourth case gives the required N,

$$14506^4 + 1^4 = (14506^2 + 5 \cdot 1^2)^2 - 1^2 \cdot 45872^2$$
$$= 210424061^2 - 45872^2 = 210378169 \cdot 210469913.$$

A table, prepared by the SOLVER (not yet published) of the roots of the congruence $x^4 + 1 \equiv 0 \pmod{p}$, shows that N has no factors $< 10{,}000$. With this start, LAURANCE'S process now shows *both* these large factors of N to be *prime*.

14854. (Rev. T. WIGGINS, B.A.)—Take a point P within a triangle ABC. Denote the lines PA, PB, PC by x, y, z, and the angles PAC, PBA, PCB by α, β, γ respectively; show that (1)

$$\frac{ax}{\sin(C-\gamma+\beta)} = \frac{by}{\sin(A-\alpha+\gamma)} = \frac{cz}{\sin(B-\beta+\alpha)};$$

(2) if $C-\gamma+\beta = 90°$,

then $a^2x^2 = b^2y^2 + c^2z^2$.

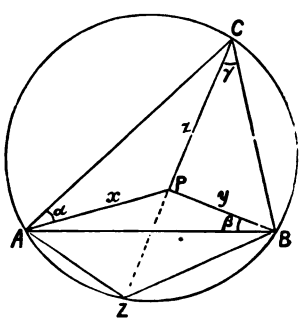
Solutions (I.) *by* T. FRAZER; (II.) *by* CONSTANCE I. MARKS, B.A.

(I.) 1. $\frac{x}{PZ} = \frac{\sin B}{\sin(A-\alpha+\gamma)}$,

$\frac{y}{PZ} = \frac{\sin A}{\sin(C-\gamma+\beta)}$;

hence $\frac{x}{y} = \frac{\sin B}{\sin A} \frac{\sin(C-\gamma+\beta)}{\sin(A-\alpha+\gamma)}$;

$\therefore \quad \frac{ax}{by} = \frac{\sin(C-\gamma+\beta)}{\sin(A-\alpha+\gamma)}$.



Similarly, by producing BP to meet the circle,

$$\frac{ax}{cz} = \frac{\sin(C-\gamma+\beta)}{\sin(B-\beta+\alpha)}.$$

Hence $$\frac{ax}{\sin(C-\gamma+\beta)} = \frac{by}{\sin(A-\alpha+\gamma)} = \frac{cz}{\sin(B-\beta+\alpha)}.$$

2. $$\frac{b^2y^2 + c^2z^2}{a^2x^2} = \sin^2(B-\beta+\alpha) + \sin^2(A-\alpha+\gamma) \quad [\text{since } C-\gamma+\beta = 90°]$$
$$= \sin^2(180°-\beta-C-A+\alpha) + \sin^2(A-\alpha+\gamma)$$
$$= \sin^2(180°-90°-\gamma-A+\alpha) + \sin^2(A-\alpha+\gamma)$$
$$= \sin^2[90°-(A-\alpha+\gamma)] + \sin^2(A-\alpha+\gamma) = 1.$$

Hence $$a^2x^2 = b^2y^2 + c^2z^2.$$

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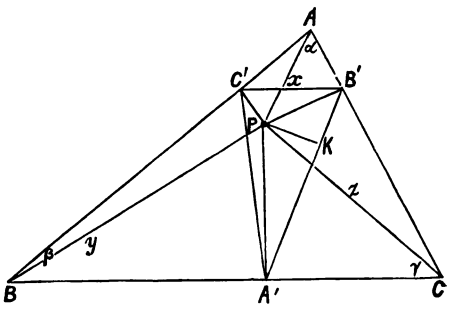
(II.) Let A′B′C′ be a new triangle formed by joining the feet of the perpendiculars from P on BC, CA, AB.

Draw PK perpendicular to A′B′.

The quadrilaterals PA′B′C, PC′A′B, PB′C′A are cyclic.

Hence it follows that the angles A′, B′, C′ are respectively equal to $C-\gamma+\beta$, $A-\alpha+\gamma$, $B-\beta+\alpha$, and that $z \,.\, PK = PA' \,.\, PB'$. Now



$$A'B' \,.\, PK = PA' \,.\, PB' \sin C = z \,.\, PK \sin C,$$

whence $$A'B' = z \sin C.$$

Similarly, it may be shown that

$$C'A' = y \sin B \quad \text{and} \quad B'C' = x \sin A.$$

Therefore, since the sides of any triangle are proportional to the sines of the opposite angles,

$$\sin(A-\alpha+\gamma) : \sin(B-\beta+\alpha) : \sin(C-\gamma+\beta) = by : cz : ax,$$

or $$\frac{ax}{\sin(C-\gamma+\beta)} = \frac{by}{\sin(A-\alpha+\gamma)} = \frac{cz}{\sin(B-\beta+\alpha)}.$$

If $$C-\gamma+\beta = 90°,$$

$$\begin{aligned} b^2y^2 + c^2z^2 &= a^2x^2[\sin^2(A-\alpha+\gamma) + \sin^2(B-\beta+\alpha)] \\ &= \tfrac{1}{2}a^2x^2\{2-[\cos 2(A-\alpha+\gamma) + \cos 2(B-\beta+\alpha)]\} \\ &= a^2x^2[1-\cos(A+B-\beta+\gamma)\cos(B-A+2\alpha-\beta-\gamma)] \\ &= a^2x^2 \quad [\text{since } \cos(A+B-\beta+\gamma) = \cos\tfrac{1}{2}\pi = 0]. \end{aligned}$$

3386. (J. J. Walker, F.R.S.) — If normals to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ be drawn from any point on the curve

$$(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2c^4x^2y^2 = 0,$$

prove that they form an harmonic pencil.

Solution by Professor Nanson.

The magical equation of the normal is

$$y - mx = c^2m(a^2 + m^2b^2)^{\frac{1}{2}},$$

and, rationalizing, we get for the m of a normal through xy the quartic

$$b^2x^2m^4 - 2b^2xym^3 + (a^2x^2 + b^2y^2 - c^4)m^2 - 2a^2xym + a^2y^2 = 0,$$

and the condition $J = 0$ for harmonicism reduces to

$$(a^2x^2 + b^2y^2 - c^4)^3 + 54a^2b^2c^4x^2y^2.$$

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8874. (Professor GENESE, M.A.)—The locus of the centres of sections of the conicoid $f(xyz) = 0$ by planes containing the axis of z is the conic determined by $df/dz = 0$, $x(df/dx) + y(df/dy) = 0$.

Solution by C. A. LAISANT.

Let $y = mx$ ..... (1),

the equation of a particular plane containing OZ. Then $f(x, mx, z) = 0$ is the equation of the projection, on OXZ, of the variable conic.

The centre is given by (1) and

$$\partial f/\partial x + m(\partial f/\partial y) = 0, \quad \partial f/\partial z = 0 \text{ ..... (2, 3).}$$

Hence, eliminating m,

$$\partial f/\partial z = 0, \quad x(\partial f/\partial x) + y(\partial f/\partial y) = 0.$$

Note.—We have here: (i.) the conjugate plane of the direction OZ; (ii.) the conicoid locus of the middle points of the chords passing by O; the intersections give the answer.

9541. (Professor BORDAGE.)—Construct a quadrilateral, knowing the projections of the point of intersection of the diagonals upon the four sides.

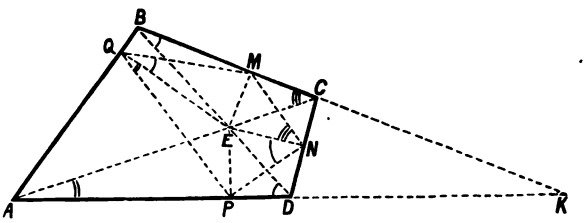
Solution by Professor IGNACIO BEYENS, Lt.-Col. du Génie.

Soient M, N, P, Q les projections du rencontre des diagonaux sur les côtés du quadrilatère demandé BCDA; on aura (voyez la figure)

$$MEP = MEN + PEN,$$

$$MEN = 180° - C = 180° - (MCE + ECN) = 180° - MCE - ECN,$$

$$PEN = 180° - D = 180° - (PDE + NDE) = 180° - PDE - NDE;$$



mais les quadrilatères EMCN et ENDP étant inscriptibles, on aura MCE = MNE et PDE = PNE, et par conséquence on a

$$MEN = 180° - MNE - ECN, \quad PEN = 180° - PNE - NDE,$$

d'où $MEP = 360° - MNP - ACD - CDB$ .....(1).

Mais, K désignant le rencontre des côtés BC et AD, donne

$$MEP = 180° - BKA \text{ .....(2);}$$

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la relation (1) donnera aussi

$$\begin{aligned} MEP &= (180°-ACD)+(180°-CDB)-MNP \\ &= ACB+DCK+ADB+CDK-MNP \\ &= ACB+180°-BKA+ADB-MNP, \end{aligned}$$

ou $$\begin{aligned} MEP &= BKA+EAP+BKA+EBC+180°-BKA-MNP \\ &= EQP+BKA+EQM+180°-MNP, \end{aligned}$$

ou $$MEP = MQP+BKA+180°-MNP \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

et, additionant les relations (2) et (3), on aura

$$2MEP = 180°-BKA+MQP+BKA+180°-MNP,$$

d'où $$MEP = 180°-\tfrac{1}{2}(MNP-MQP);$$

donc le point E est sur l'arc capable de l'angle $180°-\frac{1}{2}(MNP-MQP)$ décrit sur la droite MP. De la même manière devra être situé sur un autre arc décrit sur QN capable de l'angle $180°-\frac{1}{2}(NMQ-NPQ)$, leur intersection sera le point de rencontre des diagonaux on joindra avec M, N, P, Q, et menant par ces points les perpendiculaires correspondants, on obtiendra le quadrilatère demandé.

14941. (R. Knowles.)—Tangents are drawn to an ellipse, whose major axis is AA′, from any point in the ordinate from the focus nearest A; tangents from A, A′ meet these in MN, M′N′, respectively. Prove that $MN : M'N' = 1-e : 1+e$, e being the eccentricity of the ellipse.

Solution by A. W. Poole.

Let tangents NN′, MM′ be

$x/a\cos\phi + y/b\sin\phi = 1,$

$x/a\cos\theta + y/b\sin\theta = 1.$

These meet where

$x = a[\cos\frac{1}{2}(\theta+\phi)/\cos\frac{1}{2}(\theta-\phi)].$

But PS is the line $x = ae$; therefore

$\cos\frac{1}{2}(\theta+\phi) = e\cos\frac{1}{2}(\theta-\phi).$

Tangents at A, A′ are $x = \pm a$; therefore AM is got from

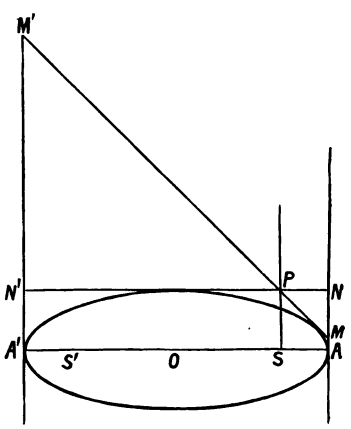
$x/a\cos\theta + y/b\sin\theta = 1$

by putting $x = a$; therefore

$$AM = \frac{b(1-\cos\theta)}{\sin\theta};$$

therefore

$$MN = b\left(\frac{1-\cos\phi}{\sin\phi} - \frac{1-\cos\theta}{\sin\theta}\right).$$



Also $$M'N' = b\left(\frac{1+\cos\theta}{\sin\theta} - \frac{1+\cos\phi}{\sin\phi}\right);$$

therefore $$\frac{MN}{M'N'} = \frac{\sin\theta-\sin\phi-\sin(\theta-\phi)}{\sin\phi-\sin\theta-\sin(\theta-\phi)}$$

$$= \frac{\cos\frac{1}{2}(\theta+\phi)-\cos\frac{1}{2}(\theta-\phi)}{-\cos\frac{1}{2}(\theta+\phi)-\cos\frac{1}{2}(\theta-\phi)} = \frac{e-1}{-e-1} = \frac{1-e}{1+e}.$$

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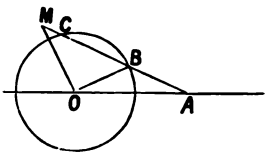
14932. (Professor Cochez.) — On donne un cercle $x^2+y^2=R^2$, et un point fixe A, par lequel on mène une sécante ABC. On construit le rayon OB et la perpendiculaire OM à OB rencontrante la sécante en M. Lieu de M.



Solution by A. W. Poole.

Let B be the point $(R\cos\theta, R\sin\theta)$, and let $OA = a$. Then equation of OB is $x/\cos\theta = y/\sin\theta$.

Therefore equation of OM is $x\cos\theta + y\sin\theta = 0$ ..... (1).

AB is the line $(x-a)/(R\cos\theta - a) = y/(R\sin\theta)$,

or $yR\cos\theta + R(a-x)\sin\theta - ay = 0$ ..... (2).

Therefore, for locus of M, we eliminate θ between (1) and (2); we have

$$\cos\theta/(-ay) = \sin\theta/axy = 1/\{R[ax-(x^2+y^2)]\}.$$

Therefore locus of M is $a^2y^2(x^2+y^2) = R^2[ax-(x^2+y^2)]^2$.

14945. (W. C. Stanham.)—Solve in positive integers $x^3-y^3=17z^3$.

Solution by Lt.-Col. Allan Cunningham, R.E.

Here $(x-y)(x^2+xy+y^2) = 17z^3$ ..... (1),

and (x^2+xy+y^2) is known to contain no divisors of form $(3i+2)$; so that 17 is a non-divisor thereof and must therefore be a factor of $(x-y)$; so that $x-y = 17\lambda$. Substituting for $(x-y)$ and dividing by 17 gives

$$\lambda(289\lambda^2+3xy) = z^3 \text{ ..... (2).}$$

Hence z must contain λ, or $z = \lambda\zeta$, suppose. Therefore

$$289\lambda^2+3xy = \lambda^2\zeta^3 \text{ ..... (3).}$$

Hence xy must contain λ^2; hence either

$$\text{(i.) } x = \lambda^2\xi, \text{ or (ii.) } y = \lambda^2\eta, \text{ or (iii.) } x = \lambda\xi,\ y = \lambda\eta \text{ ..... (4).}$$

But, since $x-y = 17\lambda$, (i.) requires $y = \lambda\eta$ and (ii.) requires $x = \lambda\xi$ in addition. Thus each case requires $x = \lambda\xi$, $y = \lambda\eta$, $z = \lambda\zeta$; and the original equation is divisible by λ^3. It suffices to exclude this case, and there will be *no loss* of generality in taking $\lambda = 1$.

Hereby equation (2) gives $xy = \frac{1}{3}(z^3-289)$ ..... (5).

Hence z must be of form $3\zeta+1$ and > 6 ..... (6).

Combining (5) with $x-y = 17$ gives

$$(x+y)^2 = \tfrac{1}{3}(4z^3-289) \text{ to be a square ..... (7).}$$

On trial $z = 7$ gives $xy = 18$, $x+y = 19$, $x = 18$, $y = 1$; and there is no other solution until $z > 25$.

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14743. (Professor N. BHATTACHARYYA.)—Find the envelope of the chord of intersection of the ellipse $x^2/a^2 + y^2/b^2 = 1$, and the circle which touches it and passes through its centre. Show that the first positive pedal of the envelope is the ellipse $x^2/b^2 + y^2/a^2 = (1-e^2)/e^4$.

Solutions by A. M. NESBITT, M.A.

The equation of any conic touching $x^2/a^2 + y^2/b^2 = 1$ at the point ϕ, and passing through centre is

$$x^2/a^2 + y^2/b^2 - 1 = (x\cos\phi/a + y\sin\phi/b - 1)(lx + my - 1),$$

and for this to be a circle we must have

$$l = -[ae^2\cos\phi/(b^2\cos^2\phi + a^2\sin^2\phi)],$$

$$m = (a^2e^2/b)\,[\sin\phi/(b^2\cos^2\phi + a^2\sin^2\phi)],$$

and we have to find the envelope of

$$a^2e^2y\sin\phi - abe^2x\cos\phi + b\,(b^2\cos^2\phi + a^2\sin^2\phi) = 0.$$

N.B.—The first positive pedal of this envelope is easily got by solving the above equation with $ax\sin\phi + by\cos\phi = 0$, for, if we write

$$\mathrm{M} = \sqrt{(b^2y^2 + a^2x^2)},$$

we get at once $ya^2e^2\,(by/\mathrm{M}) + xabe^2\,(ax/\mathrm{M}) + b\,[a^2b^2\,(x^2+y^2)]/\mathrm{M}^2 = 0$, yielding, on division by x^2+y^2,

$$e^2\mathrm{M} = b^2 \quad \text{or} \quad x^2/b^2 + y^2/a^2 = b^2/a^2e^4 = (1-e^2)/e^4.$$

Now the p, r equation of this is

$$a^2\beta^2/p^2 + r^2 = a^2 + \beta^2 \text{ (where } \alpha, \beta \text{ denote its semi-axes)},$$

or

$$r^2/p = (r^2/\alpha\beta)\sqrt{(a^2+\beta^2-r^2)};$$

and therefore the first negative pedal (which is the envelope we are in search of) has for equation $r = (p^2/\alpha\beta)\sqrt{(a^2+\beta^2-p^2)}$,

where $\quad a^2+\beta^2 = [(a^2+b^2)(1-e^2)]/e^4, \quad \alpha\beta = [ab\,(1-e^2)]/e^4.$

14631. (Professor SANJÁNA, M.A.)—A, B, C being the angles of a triangle, prove that

$$\cos A\sin^3 A\sin^4(B-C) + \cos B\sin^3 B\sin^4(C-A) + \cos C\sin^3 C\sin^4(A-B)$$
$$= \sin A\sin B\sin C\left\{\sin^2(A-B)\sin^2(C-A) + \sin^2(B-C)\sin^2(A-B) + \sin^2(C-A)\sin^2(B-C)\right\}.$$

Solution by C. E. YOUNGMAN, M.A.

The areal equation of the nine-point circle is

$$\Sigma a^2\sin B\sin C\cos A = \Sigma\beta\gamma\sin^2 A.$$

Hence the geometrical interpretation of the question is that the nine-point circle passes through the point

$$\sin^2 A\sin^2(B-C), \quad \sin^2 B\sin^2(C-A), \quad \sin^2 C\sin^2(A-B);$$

which is true, for this point is the centre of the rectangular (KIEPERT) hyperbola $\Sigma\beta\gamma\sin A\sin(B-C) = 0$.

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14692. (Robert W. D. Christie.)—Find perfectly general expressions for the sum of any n integral squares equal to the sum of n others when the sums of their roots are equal, also $(n>2)$: *e.g.*, $n=3$,

$$(a+b+c)^2+(a+d+e)^2+(a+c-b-d)^2 = (a+c-b)^2+(a+c-d)^2+(a+b+c+d)^2.$$

Solution by the Proposer.

Suppose we want five squares to equal the sum of five others, the sums of their roots being at the same time equal. Let the first four $(n-1)$ letters be constant, and change final letters thus

$$\underset{(1)}{(a+b+c+d+e)^2}+\underset{(2)}{(a+b+c+d+f)^2}+\underset{(3)}{(a+b+c+d+g)^2} + \underset{(4)}{(a+b+c+d+h)^2}+\underset{(5)\ (6)\ (7)\ (8)}{(a+b+c+d-e-f-g-h)^2},$$

and for the dexter

$$\equiv \underset{(5)}{(a+b+c+d-e)^2}+\underset{(6)}{(a+b+c+d-f)^2}+\underset{(7)}{(a+b+c+d-g)^2} + \underset{(8)}{(a+b+c+d-h)^2}+\underset{(1)\ (2)\ (3)\ (4)}{(a+b+c+d+e+f+g+h)^2}.$$

(1) *Evidently* the sums of the roots are equal, being $n\Sigma a$.

(2) In the development of the expressions the squares-sums are equal also.

(3) It remains to prove that the *products* are equal also. To do this, take any of the numbers beneath the final letter, say (1) as a sample of each. Now evidently the product $e(a+b+c+d)$ on the left $=e(a+b+c+d)$ on the right, and similarly with the other numbers and products. The equation clearly is an identity, and, as it fulfils the two requisite conditions stated *supra*, we may add n to each square. It will be convenient, in order to utilize the squares, to take $n=e+f+g+h$; thus finally

$$\begin{aligned}&(a+b+c+d+2e+f+g+h)^2+(a+b+c+d+e+2f+g+h)^2\\ &+(a+b+c+d+e+f+2g+h)^2+(a+b+c+d+e+f+g+2h)^2\\ &+(a+b+c+d)^2\\ \equiv\ &(a+b+c+d+f+g+h)^2+(a+b+c+d+e+g+h)^2\\ &+(a+b+c+d+e+f+h)^2+(a+b+c+d+e+f+g)^2\\ &+(a+b+c+d+2e+2f+2g+2h)^2,\end{aligned}$$

which may be simplified to suit requirements.

All the quantities are arbitrary, the process is perfectly general, and n may be added to each square, the result being still an identity. (See Quest. 14957.)

14537. (R. Chartres.)—A point P is taken at random in a polygon ABCD ... of n sides. What is the probability that, if n other random points be taken, one shall lie in each of the triangles APB, BPC, &c. ?

Solution by the Proposer.

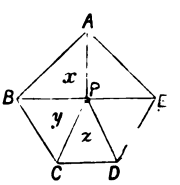
Let the area of the polygon be unity; then

$$1^n=(x+y+z+\ldots)^n,$$

which contains $(2n-1)!/[n!\,(n-1)!]$ terms whose mean values are the same; therefore the probability

$$= n!\,xyz\ldots = 1+(2n-1)!/[n!\,(n-1)!]$$
$$= [n!\,(n-1)!]/(2n-1)!.$$

Cor.—If $n=3$, the probability $=\frac{1}{10}$.



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14055. (G. H. HARDY.)—Express

$$\int_0^{\frac{1}{2}\pi} \log\sin x \sqrt{(\sin x)}\, dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\log\sin x}{\sqrt{(\sin x)}}\, dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\log\sin x}{\sqrt{(1+\sin^2 x)}}\, dx,$$

$$\int_0^{\pi} \frac{dx}{(\sec\frac{1}{2}x + \tan\frac{1}{2}x)^m \sqrt{(\sin x)}} \quad (m \text{ a positive integer}),$$

$$\int_0^{\pi} \frac{\log(\sec\frac{1}{2}x + \tan\frac{1}{2}x)}{\sqrt{(\sin x)}}\, dx, \quad \text{and} \quad \int_0^{\infty} \frac{x}{\sqrt{(\sinh x)}}\, dx$$

as multiples of $[\Gamma(\frac{1}{4})]^{\pm 2}$; and prove that

$$\int_0^{\frac{1}{2}\pi} \sin x\, (\log\sin x)^2\, dx = (\log 2 - 1)^2 + 1 - \tfrac{1}{12}\pi^2.$$

[The last integral is given in Quest. 8423 (D. EDWARDES); another proof is asked for.]

Solution by the PROPOSER.

(*a*) Let $$f(n, p) = \int_0^{\frac{1}{2}\pi} \sin^n x\, (\log\sin x)^p\, dx \quad (-1 < n,\ -1 < p).$$

Then it is easy to prove that (WOLSTENHOLME'S *Problems*, 1918, 51), if

$-1 < n < 1$, $$f(n, 0) f(-n, 0) = (2\pi/n) \tan\tfrac{1}{2}n\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and, if $0 < n$, $$f(n, 0) f(n-1, 0) = 2\pi/n \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Differentiating these equations with respect to n,

$$f(n, 1) f(-n, 0) - f(n, 0) f(-n, 1) = -(2\pi/n^2) \tan\tfrac{1}{2}n\pi + (\pi^2/n) \sec^2\tfrac{1}{2}n\pi,$$
$$f(n, 1) f(n-1, 0) + f(n, 0) f(n-1, 1) = -(2\pi/n^2).$$

If $n = \frac{1}{2}$, $$f(\tfrac{1}{2}, 1) f(-\tfrac{1}{2}, 0) - f(\tfrac{1}{2}, 0) f(-\tfrac{1}{2}, 1) = -8\pi + 4\pi^2,$$
$$f(\tfrac{1}{2}, 1) f(-\tfrac{1}{2}, 0) + f(\tfrac{1}{2}, 0) f(-\tfrac{1}{2}, 1) = -8\pi.$$

Therefore $$f(\tfrac{1}{2}, 1) = \frac{2\pi(\pi-4)}{f(-\frac{1}{2}, 0)}, \quad f(-\tfrac{1}{2}, 1) = -\frac{2\pi^2}{f(\frac{1}{2}, 0)}.$$

Now $$f(-\tfrac{1}{2}, 0) = \sqrt{\pi}\, \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{[\Gamma(\frac{1}{4})]^2}{\sqrt{(2\pi)}},$$

$$f(\tfrac{1}{2}, 0) = \sqrt{\pi}\, \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{4}{4})} = \frac{4\pi\sqrt{(2\pi)}}{[\Gamma(\frac{1}{4})]^2}.$$

Hence $$\int_0^{\frac{1}{2}\pi} \sqrt{(\sin x)} \log\sin x\, dx = \frac{(\pi-4)(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2},$$

$$\int_0^{\frac{1}{2}\pi} \frac{\log\sin x\, dx}{\sqrt{(\sin x)}} = -2^{-\frac{3}{2}} \sqrt{\pi}\, [\Gamma(\tfrac{1}{4})]^2.$$

(*b*) Suppose $$\operatorname{cosec} x = \sec\tfrac{1}{2}\theta + \tan\tfrac{1}{2}\theta.$$

Then $$\frac{dx}{\sqrt{(\sin x)}} = -\frac{(\sin x)}{\cos x} \frac{1 + \sin\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}\, d\theta = -\tfrac{1}{2} \frac{\sqrt{(\sin x)}}{\cos x} \frac{d\theta}{\cos\frac{1}{2}\theta}$$
$$= -\tfrac{1}{2} \sec\tfrac{1}{2}\theta \sqrt{\left(\frac{\sec\frac{1}{2}\theta + \tan\frac{1}{2}\theta}{(\sec\frac{1}{2}\theta + \tan\frac{1}{2}\theta)^2 - 1}\right)}\, d\theta = -\tfrac{1}{2} \frac{d\theta}{\sqrt{(\sin\theta)}}.$$

As x goes from 0 to $\frac{1}{2}\pi$, θ goes from π to 0. Hence

$$\int_0^{\pi} \phi(\sec\tfrac{1}{2}\theta + \tan\tfrac{1}{2}\theta) \frac{d\theta}{\sqrt{(\sin\theta)}} = 2\int_0^{\pi} \phi(\operatorname{cosec} x) \frac{dx}{\sqrt{(\sin x)}}.$$

This formula gives

$$\int_0^{\pi} \frac{\log(\sec\frac{1}{2}x + \tan\frac{1}{2}x)}{\sqrt{(\sin x)}}\, dx = -2\int_0^{\frac{1}{2}\pi} \frac{\log\sin x}{\sqrt{(\sin x)}}\, dx = 2^{\frac{1}{2}} \sqrt{\pi}\, [\Gamma(\tfrac{1}{4})]^2.$$

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And $$\int_0^{\pi} \frac{(\sec \frac{1}{4}x + \tan \frac{1}{4}x)^{-m}}{\sqrt{(\sin x)}}\,dx = -2\int_0^{\frac{1}{2}\pi} (\sin x)^{m-\frac{1}{2}}\,dx;$$

which is easily expressed in terms of $[\Gamma(\frac{1}{4})]^2$ when m is an integer.

(*c*) If we put $\sin x = e^{-u}$, $dx = -\dfrac{e^{-u}\,du}{\sqrt{(1-e^{-2u})}}$,

$$\int_0^{\frac{1}{2}\pi} \frac{\log \sin x}{\sqrt{(\sin x)}}\,dx = -\int_0^{\infty} \frac{ue^{-\frac{1}{2}u}\,du}{\sqrt{(1-e^{-2u})}} = -\frac{1}{\sqrt{2}}\int_0^{\infty} \frac{u\,du}{\sqrt{(\sinh u)}}.$$

If we put $\sin x = \sin^2 y$, $dx = \dfrac{2\sin y\,dy}{\sqrt{(1+\sin^2 y)}}$,

$$\int_0^{\frac{1}{2}\pi} \frac{\log \sin x\,dx}{\sqrt{(\sin x)}} = 4\int_0^{\frac{1}{2}\pi} \frac{\log \sin y\,dy}{\sqrt{(1+\sin^2 y)}}.$$

(*d*) We may observe that, if

$$\int_0^{\infty} e^{-x^4}dx = k, \quad ke^{-a^4} = \int_0^{\infty} e^{-a^4(1+x^4)}\,a\,dx,$$

$$k^2 = \int_0^{\infty} da \int_0^{\infty} e^{-a^4(1+x^4)}\,a\,dx = \tfrac{1}{2}\int_0^{\infty} dx \int_0^{\infty} e^{-a^4(1+x^4)}\,da$$

$$= \tfrac{1}{4}\sqrt{\pi}\int_0^{\infty} \frac{dx}{\sqrt{(1+x^4)}};$$

so that $$[\Gamma(\tfrac{1}{4})]^2 = 4\sqrt{\pi}\int_0^{\infty} \frac{dx}{\sqrt{(1+x^4)}}.$$

Thus, *e.g.*, $$\int_0^{\frac{1}{2}\pi} \frac{\log \sin x\,dx}{\sqrt{(\sin x)}} = -\pi\sqrt{2}\int_0^{\infty} \frac{dx}{\sqrt{(1+x^4)}}.$$

14405. (H. MacColl, B.A.)—To make sense of the implication, "If it is x that A and B are y, it is y that A and B are x," one of the three words, *true*, *certain*, *probable*, is taken at random and substituted for x; then one of the same three words is taken at random and substituted for y. What is the chance that the implication, thus rendered intelligible, will be a *formal certainty*? (I find *two-thirds*; but my reasoning involves one or two controvertible points, and I should be glad to see other solutions.) See *Reprint*, Vol. LXXIV., p. 36.

Solution by the Proposer.

Using the notation already explained in the *Educational Times*, let $\phi(x, y)$ denote the given implication $(A^yB^y)^x : (A^xB^x)^y$; and let the symbols τ, ϵ, p respectively denote the words *true*, *certain*, *probable*. The nine possible cases are $\phi(\tau, \tau)$, $\phi(\epsilon, \epsilon)$, $\phi(p, p)$, $\phi(\tau, \epsilon)$, $\phi(\epsilon, \tau)$, $\phi(p, \tau)$, $\phi(\tau, p)$, $\phi(p, \epsilon)$, $\phi(\epsilon, p)$. That the first five are formal certainties is evident. The sixth (which may be read, "If it is *probable* that A and B are *true*, it is *true* that A and B are *probable*") may be proved a formal certainty as follows:—

$$\phi(p, \tau) = (A^{\tau}B^{\tau})^p : (A^pB^p)^{\tau} = (AB)^p : A^pB$$

the predicative exponent τ being understood when not marked. Now,

$$(AB)^p : \left(\frac{AB}{\epsilon} > \tfrac{1}{2}\right) : \left(\frac{A}{\epsilon}\cdot\frac{B}{A} > \tfrac{1}{2}\right) : \left(\frac{A}{\epsilon} > \tfrac{1}{2}\right);$$

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for, since the chance $\frac{B}{A}$ cannot exceed 1, the chance $\frac{A}{\epsilon}$ cannot be less than $\frac{A}{\epsilon}.\frac{B}{A}$. But the statement $\left(\frac{A}{\epsilon} > \frac{1}{2}\right)$ is synonymous with A^p, so that we have proved $(AB)^p : A^p$. Similarly we prove $(AB)^p : B^p$; and these two proved implications combined make $(AB)^p : A^pB^p$, which is synonymous with $\phi(p, \tau)$. The first six of the nine cases are therefore formal certainties. The remaining three cases, however, are *not universally true;* and to prove this we have only to find a single case of failure for each.

Take $\phi(\tau, p)$, which asserts that, "If it is *true* that A and B are *probable*, it is *probable* that A and B are *true*." Suppose A and B to be *independent*, and that the chance of each is *two-thirds*. Here it is true that A and B are probable (since the chance of each is greater than *one-half*); but, since (by reason of their independence) the chance that *both* are true is *four-ninths*, and therefore less than *one-half*, it is *not probable* that A and B are *true*.

Next, take $\phi(p, \epsilon)$, which asserts that, "If it is *probable* that A and B are *certain*, it is *certain* that A and B are *probable*." Let there be five equally possible hypotheses, H_1, H_2, H_3, &c., of which one must be, and only one can be, true at the same time; so that the chance of each is *one-fifth*. Let each of the three H_1, H_2, H_3 lead necessarily to the conclusion that the chance of each of the statements A and B being true is *unity;* let H_4 lead to the conclusion that this chance is *two-thirds;* while H_5 implies that it is *one-seventh*. To find the chance of $A^\epsilon B^\epsilon$, that A and B are *certain*, we have $A^\epsilon B^\epsilon = A^\epsilon B^\epsilon (H_1 + H_2 + H_3 + H_4 + H_5)$;

$$\text{therefore} \quad \frac{A^\epsilon B^\epsilon}{\epsilon} = \frac{H_1}{\epsilon}.\frac{A^\epsilon B^\epsilon}{H_1} + \frac{H_2}{\epsilon}.\frac{A^\epsilon B^\epsilon}{H_2} + \&c. = \tfrac{1}{5}\frac{A^\epsilon B^\epsilon}{H_1} + \tfrac{1}{5}\frac{A^\epsilon B^\epsilon}{H_2} + \&c.$$
$$= \tfrac{1}{5}(1 + 1 + 1 + 0 + 0) = \tfrac{3}{5}.$$

Hence, it is *probable* that A and B are *certain*. To find the chance A^pB^p (that A and B are *probable*) we similarly get

$$\frac{A^pB^p}{\epsilon} = \tfrac{1}{5}\frac{A^pB^p}{H_1} + \tfrac{1}{5}\frac{A^pB^p}{H_2} + \&c. = \tfrac{1}{5}(1 + 1 + 1 + 1 + 0) = \tfrac{4}{5}.$$

Hence, it is *not certain* (though it is *probable*) that A and B are *probable*. In this case therefore $\phi(p, \epsilon)$ fails; so that it is *not universally true*.

Lastly, take $\phi(\epsilon, p)$, which asserts that, "If it is *certain* that A and B are *probable*, it is *probable* that A and B are *certain*." Let there be two possible and mutually exclusive hypotheses, H_1 and H_2, the chance of each being *one-half*, and let each imply that A and B are *probable but not certain*, the chance of A and B being each *two-thirds*. We first get

$$\frac{A^pB^p}{\epsilon} = \frac{A^pB^p(H_1 + H_2)}{\epsilon} = \frac{H_1}{\epsilon}.\frac{A^pB^p}{H_1} + \frac{H_2}{\epsilon}.\frac{A^pB}{H_2} = (\tfrac{1}{2} \times 1) + (\tfrac{1}{2} \times 1) = 1;$$

so that it is *certain* that A and B are *probable*. Similarly, we get

$$\frac{A^\epsilon B^\epsilon}{\epsilon} = \frac{H_1}{\epsilon}.\frac{A^\epsilon B^\epsilon}{H_1} + \frac{H_2}{\epsilon}.\frac{A^\epsilon B^\epsilon}{H_2} = (\tfrac{1}{2} \times 0) + (\tfrac{1}{2} \times 0) = 0;$$

so that it is *impossible*, and therefore, *a fortiori*, *not probable*, that A and B are *certain*. This, consequently, is a case of failure which shows that $\phi(\epsilon, p)$ is *not true universally*.

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Hence, there are *six* cases in which $\phi(x, y)$ is a *formal certainty*, and *three* cases in which it is not; so that the required chance is *two-thirds*.

Note.—It will be observed that I draw a wide distinction between the words *true* and *certain*. I call a statement *certain* when it necessarily follows from data *assumed throughout the argument*, including definitions of symbols, tacitly assumed axioms, &c. A *true* statement (true in a particular case) is not necessarily *certain* (true in all admissible cases); but a *certain* statement is necessarily *true*. Thus $(A^e : A)$ is a formal certainty; $(A : A^e)$ is not.

13382. (Professor NANSON.)—If u, v are n-ary quadrics in $x_1, \ldots, x_n$, and $I_m = \int \ldots \int v^m dx_1, \ldots, dx_n$, the limits being given by $0 \leqslant u \leqslant 1$, u being definite and positive, prove that $\delta I_m = -(\frac{1}{2}n + m + 1) I_{m+1}$, where

$$\delta = a'(d/da) + b'(d/db) + \ldots,$$

$a, b, \ldots$ being the coefficients of u, and $a', b', \ldots$ those of v. Hence express I_m in terms of the invariants of u, v, m being a positive integer.

Solution by Professor S. SIRCOM.

Let the quadrics be reduced so as to have a common self-conjugate, and write x_1^2 for ax_1^2, &c.; then

$$u = \Sigma x^2, \quad v = \Sigma\left(\frac{a_1}{a} x^2\right), \quad I_m = \frac{1}{\sqrt{(ab\ldots)}} \int \ldots \int v^m dx, \ \ldots, \ dx_n,$$

the limits being given by $0 \leqslant u \leqslant 1$. Now

$$\int \ldots \int v^m dx_1, \ldots, dx_m = \frac{2m+n+2}{2m+n} \int \ldots \int \Sigma(x^2) v^m dx_1, \ldots, dx_n,$$

which may be proved by making the polar transformation $\Sigma x^2 = r^2$, &c., and integrating from $r = 0$ to $r = 1$; whence

$$\delta I_m = -\frac{1}{\sqrt{\Delta}} \frac{2m+n+2}{2m+n} \left\{ \tfrac{1}{2} \int \ldots \int \Sigma\left(\frac{a_1}{a}\right) \Sigma(x^2) v^m dx_1, \ldots, dx_n \right.$$
$$\left. + m \int \ldots \int \Sigma\left(\frac{a_1^2}{a^2} x^2\right) \Sigma(x^2) v^{m-1} dx_1, \ldots, dx_n \right\}$$
$$= -\frac{1}{\sqrt{\Delta}} \frac{2m+n+2}{2m+n} (\tfrac{1}{2}U + mV) \quad \ldots\ldots(1),$$

$$V = \sqrt{\Delta}\, I_{m+1} + \int \ldots \int \Sigma \left\{ \left(\frac{a_1^2}{a^2} + \frac{b_1^2}{b^2}\right) x_1^2 x_2^2 \right\} v^{m-1} dx_1, \ldots, dx_n$$
$$-2 \int \ldots \int \Sigma \left(\frac{a_1 b_1}{ab} x_1^2 x_2^2\right) v^{m-1} dx_1, \ldots, dx_n \quad \ldots(2).$$

Integrating by parts successively with respect to $x_2, \ldots, x_n$, and adding the results, we have

$$(n-1) \int \ldots \int \frac{a_1}{a} x_1^2 v^m dx_1, \ldots, dx_n = \int \ldots \int \frac{a_1}{a} x_1^2 x_2 v^m dx_1, dx_2, \ldots, dx_n + \ldots$$
$$-2m \int \ldots \int \frac{a_1 b_1}{ab} x_1^2 x_2^2 v^{m-1} dx_1, \ldots, dx_n - \ldots,$$

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where the limits have still to be substituted; whence, adding all the integrals of this form,

$$\sqrt{\Delta}\,(n-1)\,I_{m+1} = \Sigma\int\ldots\int \frac{a_1}{a}\,x_1^2x_2v^m\,dx_1,\,dx_3,\,\ldots,\,dx_n$$
$$-4m\int\ldots\int\Sigma\left(\frac{a_1b_1}{ab}\,x_1^2x_2^2\right)v^{m-1}dx_1,\,\ldots,\,dx_n \quad \ldots(3).$$

Also $$U = \sqrt{\Delta}\,I_{m+1} + \int\ldots\int\Sigma\frac{a_1}{a}\,(x_2^2+\ldots+x_n^2)\,v^m dx_1,\,\ldots,\,dx_n,$$

and, integrating by parts and adding the results,

$$\tfrac{1}{2}\int\ldots\int\Sigma\frac{a_1}{a}\,(x_2^2+\ldots+x_n^2)\,v^m dx_1,\,\ldots,\,dx_n$$
$$+m\int\ldots\int\Sigma\left\{\left(\frac{a_1^2}{a^2}+\frac{b_1^2}{b^2}\right)x_1^2x_2^2\right\}v^{m-1}dx_1,\,\ldots,\,dx_n$$
$$=\tfrac{1}{2}\Sigma\int\ldots\int\frac{a_1}{a}\,x_2^2x_1v^m dx_2,\,\ldots,\,dx_n.$$

Then, from (2) and (3),

$$\tfrac{1}{2}U + mV$$
$$= (m+\tfrac{1}{2}n)\,\sqrt{\Delta}\,I_{m+1} + \tfrac{1}{2}\Sigma\int\ldots\int v^m\frac{a_1}{a}\,(x_2^2x_1dx_2 - x_1^2x_2dx_1)dx_3,\,\ldots,\,dx_n,$$

and each of the partial integrals vanishes; then (1) becomes

$$\delta I_m = -(\tfrac{1}{2}n+m+1)\,I_{m+1}.$$

Thus $$I_0 = \frac{1}{\sqrt{\Delta}}\int\ldots\int dx_1,\,\ldots,\,dx_n = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+1)\,\sqrt{\Delta}},$$

both positive and negative values being taken.

$$I_1 = -\frac{1}{\frac{1}{2}n+1}\,\delta I_0 = \frac{\pi^{\frac{1}{2}n}}{2\Gamma(\frac{1}{2}n+2)}\,\frac{\Theta}{\Delta\sqrt{\Delta}},$$

and so on, where Δ, Θ, &c., are invariants of the original quadrics.

14776. (J. Blaikie, M.A.) — Show that any number $10a+b$ is a multiple of $10n-1$, if $a+nb$ be a multiple of $10n-1$. *E.g.*,

63	is a multiple of	9,	since	$6+3$	is a multiple of	9;
351	,,	39,	,,	$35+4$	,,	39;
1157	,,	89,	,,	$115+63$	,,	89.

[Solutions so numerous that only a few typical ones can be published. —Editor.]

Solutions (I.) *by* Robert W. D. Christie *and* Lt.-Col. Allan Cunningham, R.E.; (II.) *by the* Proposer; (III.) *by* Quæro; (IV.) *by* Aletrop.

(I.) We have, by hypothesis,

$$a+nb = 10\,(a+nb) = b\,(10n\pm1) = 0 \mod 10n\pm1;$$

therefore $$10a\mp b = 0 \mod 10n\pm1.$$

Similarly, if $a-nb = 0 \mod 10n\pm1$, $10a\pm b = 0 \mod 10n\pm1$, due regard being had to the signs.

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This theorem, useful for factorization, may be extended thus:—

If $10a-b = 0 \mod (10n+m)$, $n(10a-b) = 0 \mod (10n+m)$;

also $a(10n+m) = 0 \mod (10n+m)$;

therefore $ma \pm nb = 0 \mod (10n+m) =$ any integer whatever.

(II.) Let $a + nb = (10n-1)m$.

Then $10a + b = 10(a+nb) - (10n-1)b = (10n-1)(10m-b)$

$=$ a multiple of $10n-1$.

(III.) Let $(a+nb)/(10n-1) = q$. We get $a = 10qn - q - nb$. Substituting this value of a in the fraction $(10a+b)/(10n-1)$, it becomes $10q-b$, which is evidently an integer, provided q and b are integers. This result holds good whatever the values (integral or fractional) of a and n.

(IV.) Soit β la base du système de numération. De la double identité $\beta a + b = \beta(a \pm nb) \mp b(\beta n \mp 1)$, (les signes supérieures et inférieures étant pris ensemble), on conclut, puisque les deux nombres $\beta n \mp 1$ sont premiers avec β, que la condition nécessaire et suffisante pour que $\beta a + b$ soit un multiple de $\beta n \mp 1$, c'est que $a \pm nb$ le soit aussi, respectivement.

7903. (Rev. T. C. Simmons, M.A.)—Prove that the mean value of the nth power of the distance between two points taken at random within any convex area Ω is equal to $\dfrac{2}{(n+2)(n+3)\Omega^2}\iint C^{n+3}\,dp\,d\omega$, where C is the length of a chord whose coordinates are p, ω, and n any quantity not less than -1, the integration extending throughout the area.

Solutions (I.) *by* R. Chartres; (II.) *by* J. W. Carse, B.A.

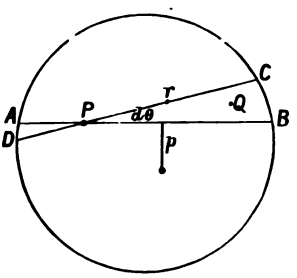
(I.) The mean value of PQ^n, if Q moves in the element PBC, is evidently $= \dfrac{2r^n}{n+2}$.

Therefore the required mean

$$= \frac{2}{n+2}\iint\left\{\int_0^C 2r^n\,(\tfrac{1}{2}r^2)\,dr\right\}dp\,d\theta \div (\text{area})^2$$

$$= \frac{2}{(n+2)(n+3)\Omega^2}\iint_0^C C^{n+3}\,dp\,d\theta.$$

This is a most valuable theorem.



AB = C.

If $n = 0$, we get the remarkable result $\iint C^3\,dp\,d\theta = 3\Omega^2$,

given in Williamson's *Integral Calculus*, p. 386.

For the circle, if $n = 1, -1$, &c., we get at once the well-known values $128r/45\pi$, $16/3\pi r$, &c.

(II.) Let one of the points A be fixed; draw through it the chord PQ = C at an inclination ω to some fixed line; put AP = r, AQ = r'; then (*cf.*

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WILLIAMSON, *Integral Calculus*, § 255) the number of cases for which the direction of the line joining A and B lies between ω and $\omega + d\omega$

$$= \int_0^r \rho^n . \rho \, d\rho \, d\theta + \int_0^{r'} \rho^n . \rho \, d\rho \, d\theta, \quad \text{where} \quad \rho = \text{AB}.$$

The sum of these integrals $= [1/(n+2)](r^{n+2} + r'^{n+2})$, $n \not< (-1)$. Now let A range over the space between PQ and a parallel chord distant dp from it, and the direction of AB from ω to $\omega + d\omega$; then the number of cases is

$$\left\{ \frac{1}{n+2} \int_0^C r^{n+2} dr + \frac{1}{n+2} \int_0^C r'^{n+2} dr' \right\} dp \, d\omega = \frac{2}{(n+2)(n+3)} C^{n+3} dp \, d\omega.$$

Integrating over the whole area, we have the total number of cases

$$= \frac{2}{(n+2)(n+3)} \iint C^{n+3} dp \, d\omega;$$

and the number of pairs of points is Ω^2. Hence the above expression for the mean value.

8139. (A. GORDON, M.A.)—Show geometrically, *by the properties of circles*, that the three middle points of the diagonals of a complete quadrilateral lie in a straight line.

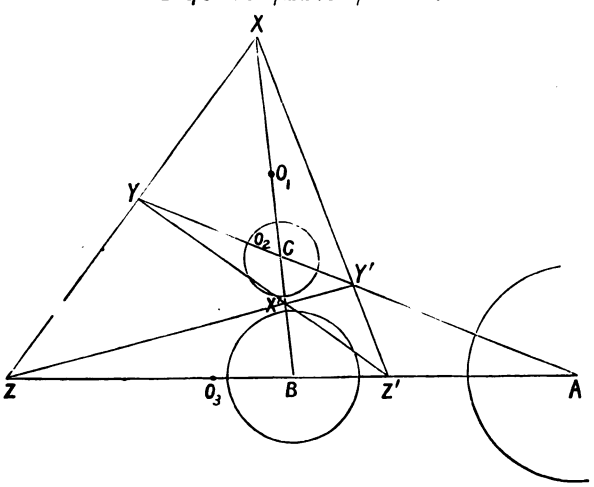
Solution by S. CONSTABLE.

Let XYX'Y' be our quadrilateral, ZZ' being third diagonal; YY', ZZ' meeting in A, &c. With centre A describe *any* circle with radius r' and, for convenience, less than AY'. With centre C describe circle of radius r''', where $r'/r''' = \text{AY/CY} = \text{AY'/CY'}$; and with centre B describe circle r'', where $r'/r'' = \text{AZ'/BZ'} = \text{AZ/BZ}$. Hence

$$r''/r''' = \text{AY/CY} . \text{BZ/AZ} = \text{BX/CX},$$

for

$$\text{BX/CX} . \text{CY/AY} . \text{AZ/BZ} = 1.$$



Hence X, X' are centres of similitude of circles (B), (C); and Y, Y'; Z, Z' are centres for (A), (C); (A), (B); respectively.

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But the circles on XX_1, YY_1, ZZ' as diameters are the three circles of similitude of (A), (B), (C); and we know from the properties of these circles that they are coaxal, and that their centres O_1, O_2, O_3 are consequently collinear.

Note.—Without using these circles, we can prove that O_1, O_2, O_3 are collinear, thus giving another very short solution of this interesting problem.

Since ZZ', XX_1, YY_1 are divided harmonically at B, A; C, B; and A, C respectively, we have

$$BO_1/CO_1 = BX^2/CX^2, \quad CO_2/AO_2 = CY^2/AY^2, \quad AO_3/BO_3 = AZ^2/BZ^2;$$

therefore

$$BO_1/CO_1 . CO_2/AO_2 . AO_3/BO_3 = BX^2/CX^2 . CY^2/AY^2 . AZ^2/BZ^2 = 1,$$

for, X, Y, Z being collinear,

$$BX/CX . CY/AY . AZ/BZ = 1.$$

Note on Quest. 14945. By the Editor.

A very interesting solution to Mr. Stanham's Problem bearing this number has already been published. The Proposer also *communicated* for solution, in positive integers, the equation $x^3+y^3=17z^3$. Both $x^3+y^3=17z^3$ and $x^3-y^3=17z^3$ have been solved in *integers*, as under, by Mr. Biddle, but the solution of the former in *positive integers* is still undiscovered.

Solution by D. Biddle.

Since $x^3-y^3=(x-y)(x^2+xy+y^2)$, we can take $x-y=17$, and find values of x and y such that x^2+xy+y^2 is a cube. This is satisfied by $x=18$, $y=1$, giving $z=7$. Similarly, since $x^3+y^3=(x+y)(x^2-xy+y^2)$, $x^3+y^3=17z^3$ is satisfied by $x=18$, $y=-1$, $z=7$. But here y is negative.

7753. (Lanivic.)—Find whether there exist even integer numbers which cannot be decomposed into a sum of two primes.

Note by Lt.-Col. Allan Cunningham, R.E.

The proof of this problem (known as Goldbach's theorem) is one of the *desiderata* of modern mathematics. Tables giving *all* the *actual solutions* of $p+q=N$ (an *even* number $\not>$ 3000), and also the *number* of solutions of the same up to $N=5000$, have been published under the title "Tafeln für das Goldbach'sche Gesetz" in *Abh. d. Kaiserl. Leop. Carol. Deutschen Akad. d. Naturforscher*, Bd. LXXII., Halle, 1897; so that the (numerical) proof is established up to $N=5000$.

A simple *mechanical* method of solution of $p+q=N$ is described at p. 9 of the introduction to above. A long strip of paper graduated into equal divisions > N in number is taken, the division marks are *numbered* where convenient—say every tenth—and the division marks corresponding to

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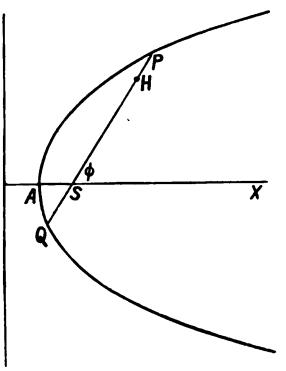
primes are *marked conspicuously*. If the strip be *doubled* so as to bring any *even* number N opposite to zero, the coincidence of any two prime division marks will give a solution of $p+q=N$.

When N exceeds a few thousand there are practical difficulties in the way of handling the long strip required, and keeping it at proper tension.

14660. (R. Tucker, M.A.)—PQ is a focal chord of a parabola (focus S), and is also the major axis of an ellipse. Find the locus of the other focus and the equation to the ellipse. If this ellipse cuts the axis in K, K′, then $1/SK+1/SK'=1/a$, where $4a$ is the latus rectum of the parabola (and also of the ellipse). Show also that the envelope of the ellipse is a confocal parabola of latus rectum $8a$, and that of the auxiliary circle is (beside the directrix) the circle $x^2+y^2-ax=2a^2$. If $\lambda=\cot\phi$ (ϕ angle made by chord with axis), then the point of contact is $x/a=-2(2\lambda^2-1)/(4\lambda^2+1)$, $y/a=-6\lambda/(4\lambda^2+1)$.

Solutions (I.) *by the* Proposer; (II.) *by* Professor Sanjána, M.A.

(I.) $\angle PSX=\phi$;



equation to the ellipse, is of form

$1/r=[1-\epsilon\cos(\theta-\phi)]/[a(1-\epsilon^2)]$...(i.);

$PQ=4a\operatorname{cosec}^2\phi$, $\therefore\ a=2a\operatorname{cosec}^2\phi$,

$2a\epsilon=SH=4a\cos\phi\operatorname{cosec}^2\phi$,

$a\epsilon=2a\operatorname{cosec}^2\phi\cos\phi$;

therefore $\epsilon=\cos\phi$;

latus rectum $=2a(1-\epsilon^2)=4a$;

therefore equation is

$2a/r=1-\epsilon\cos(\theta-\phi)$

$=1-\cos\phi\cos(\theta-\phi)$...(ii.).

$0=\sin\phi\cos(\theta-\phi)-\cos\phi\sin(\theta-\phi)$

$=\sin(2\phi-\theta)$;

envelope of ellipse is therefore

$2a/r=1-\cos^2\frac{1}{2}\theta=\sin^2\frac{1}{2}\theta$;

that is $r=4a/(1-\cos\theta)$, a parabola;

locus of H, $r=4a\cos\phi\operatorname{cosec}^2\phi$, *i.e.*, $y^2=4ax$ (S the vertex),

$\theta=0$, $r_1=2a/\sin^2\phi$, $\theta=\pi$, $r_2=2a/(1+\cos^2\phi)$;

therefore $1/r_1+1/r_2=1/2a[\sin^2\phi+(1+\cos^2\phi)]=1/a$,

$\beta^2/a^2=1-\epsilon^2=\sin^2\phi$; therefore $\beta=a\sin\phi=2a\operatorname{cosec}\phi$;

Equation to auxiliary circle is

$$x^2+y^2-4ax\cot^2\phi-4ay\cot\phi=4a^2\operatorname{cosec}^2\phi,$$

$$4a\cot^2\phi(x+a)+4a\cot\phi y+4a^2-(x^2+y^2)=0;$$

for envelope $16a^2y^2=16a(x+a)[4a^2-(x^2+y^2)]$;

therefore $y^2(x+2a)=(x+a)(4a^2-x^2)$, $y^2=(x+a)(2a-x)$;

$x^2+y^2-ax=2a^2$, $(x-\frac{1}{2}a)^2+y^2=\frac{9}{4}a^2$, and $x+2a=0$,

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the directrix. If $\cot\phi = \lambda$, the circular point of contact is

$$x/a = -2(2\lambda^2-1)/(4\lambda^2+1), \quad y/a = -6\lambda/(4\lambda^2+1).$$

(II.) It is supposed that the ellipse is also confocal. If

$$SP = r, \quad \angle PSK = \theta,$$

we have $SP = 2a/(1-\cos\theta)$,

$$SP' = 2a/(1+\cos\theta);$$

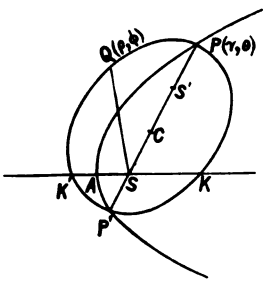
whence $SS' = 4a\cos\theta/\sin^2\theta$,

i.e., $\rho^2\sin^2\theta = 4a\rho\cos\theta$ is the equation of S'. This is the equal and similarly placed parabola $y^2 = 4ax$. As

$$PP' = 4a/\sin^2\theta,$$

the eccentricity of the ellipse is

$$SS' + PP' = \cos\theta.$$



Hence, if (ρ, ϕ) be any point Q on the ellipse, we get for its equation

$$\rho\{1-\cos\theta\cos(\phi-\theta)\} = CP(1-e^2) = 2a.$$

Transforming to Cartesians, we get

$$\rho^2 \equiv x^2+y^2 = (2a+x\cos^2\theta+y\sin\theta\cos\theta)^2.$$

Putting $y = 0$, we get for the abscissæ of K and K', $\pm x = x\cos^2\theta + 2a$; *i.e.*, $SK = 2a/(1-\cos^2\theta)$ and $SK' = 2a/(+1+\cos^2\theta)$; whence

$$1/SK + 1/SK' = 1/a.$$

Differentiating the equation of the ellipse for θ, we have

$$\rho\sin\theta\cos(\phi-\theta) - \rho\cos\theta\sin(\phi-\theta) = 0,$$

whence $\sin(2\theta-\phi) = 0$, or $\phi = 2\theta$, the point where the ellipse touches its envelope. To find this envelope, we put $\theta = \frac{1}{2}\phi$ in the equation; thus $\rho(1-\cos^2\frac{1}{2}\phi) = 2a$; therefore $2\rho\sin^2\frac{1}{2}\phi = 4a$; therefore $\rho(1-\cos\phi) = 4a$, a parabola with $8a$ for latus rectum.

Finally, the coordinates of C are $2a\cot^2\theta$, $2a\cot\theta$, and the radius of the auxiliary circle is $2a/\sin^2\theta$; hence the equation of this circle is

$$x^2+y^2-4ax\cot^2\theta-4ay\cot\theta-4a^2\cot^2\theta-4a^2 = 0,$$

or

$$\lambda^2(4a^2+4ax)+4ay\lambda+4a^2-x^2-y^2 = 0.$$

This at once gives for the envelope $ay^2 = (a+x)(4a^2-x^2-y^2)$. As the directrix $x = -2a$ is known to be part of the envelope, we take out the factor $x+2a$, and obtain as remaining part of the envelope

$$x^2+y^2-ax-2a^2 = 0.$$

Solving the circle and its envelope, we readily get

$$x = [2a(1-2\lambda^2)]/(4\lambda^2+1), \quad y = [2a(-3\lambda)]/(4\lambda^2+1);$$

also $x = -2a$, $y = 2a\lambda$ for contact with directrix.

14925. (Salutation.)—Through a particular vertex of an irregular (but convex) n-gon draw a straight line bisecting the figure.

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Solution by JOHN PRESCOTT, B.A.

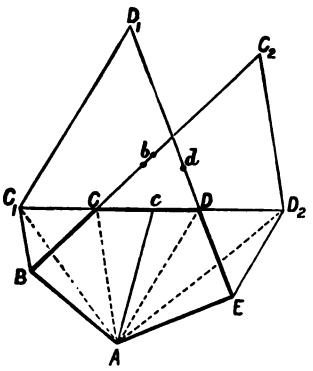
Let us take a five-sided figure for simplicity, but the method is perfectly general.

Let ABCDE be the figure, and let A be the angular point through which the bisector has to be drawn. From A draw the diagonals AC, AD.

Draw BC_1 parallel to AC to meet DC produced in C_1.

Draw C_1D_1 parallel to AD to meet ED produced in D_1.

Go the opposite direction round the figure in the same manner and call the points we get D_2, C_2.



Bisect BC_2, C_1D_1, D_1E in b, c, d respectively. Join A to whichever of the points b, c, d falls on a side of the original polygon; *i.e.*, suppose c falls between C and D; join Ac. Then Ac bisects the polygon.

For, if we join AC_1, AD_2, $\triangle AC_1C = \triangle ABC$; $\therefore \triangle AC_1c =$ figure ABCc. Also, in the same way, $\triangle AD_2c =$ figure AEDc. But $\triangle AC_1c = \triangle AD_2c$. Therefore figure ABC$c$ = AEDc.

14710. (Professor N. BHATTACHARYYA.)—From the centre of curvature at any point on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ two other normals are drawn. Show that the envelope of the line joining their feet is the curve $(x/a)^{\frac{1}{3}} + (y/b)^{\frac{1}{3}} = 1$.

Solutions (I.) *by* Professor JAN DE VRIES; (II.) *by* F. H. PEACHELL, B.A.; (III.) *by* Rev. J. CULLEN.

(I.) The feet of four concurrent normals are the intersections of the ellipse and an orthogonal hyperbola which contains the centre of the ellipse, the asymptotes being parallel to the axes of the ellipse. If $px + qy + r = 0$ represents the line joining the feet of the two normals from the centre of curvature at the point (x_1, y_1), there is an identity of the form

$$(x^2/a^2 + y^2/b^2 - 1) + \lambda(xy + Ax + By) \equiv (x_1x/a^2 + y_1y/b^2 - 1)(px + qy + r).$$

Hence $$px + qy + r \equiv x/x_1 + y/y_1 + 1.$$

Putting $x = a\cos\theta$, $y = b\sin\theta$, a point of the envelope of this line is determined by $$bx\sin\theta + ay\cos\theta + ab\sin\theta\cos\theta = 0,$$

and $$bx\cos\theta - ay\sin\theta + ab(\cos^2\theta - \sin^2\theta) = 0,$$

whence $$x = -a\cos^3\theta, \quad y = -b\sin^3\theta.$$

Therefore the envelope is the curve $(x/a)^{\frac{1}{3}} + (y/b)^{\frac{1}{3}} = 1$.

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(II.) The normals at two consecutive points on the curve intersect in the centre of curvature. But the normals at the ends of $x/a\cos\phi + y/b\sin\phi = 1$ and $x/(a\cos\phi) + y/(b\sin\phi) + 1 = 0$ meet in a point; therefore the chord whose envelope is required is of the form

$$x/(a\cos\phi) + y/(b\sin\phi) + 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Differentiating with regard to ϕ, we have

$$x/a \,.\, \sin\phi/\cos^2\phi - y/b \,.\, \cos\phi/\sin^2\phi = 0, \quad \text{or} \quad \sin^3\phi/(y/b) = \cos^3\phi/(x/a);$$

therefore $\sin\phi/(y/b)^{\frac{1}{3}} = \cos\phi/(x/a)^{\frac{1}{3}} = 1/\{(y/b)^{\frac{2}{3}} + (x/a)^{\frac{2}{3}}\}^{\frac{1}{2}}$.

Substituting in (1), we get, as equation of the envelope,

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

(III.) Let $S = x^2/a^2 + y^2/b^2 - 1$ and $S' = 2(c^2xy + b^2y'x - a^2x'y)$; then the feet of the normals lie on $S' + kS = 0$. Also $x' = a^{-1}c^2\cos^3\phi$, $y' = -b^{-1}c^2\sin^3\phi$, the coordinates of the centre of curvature. Now, since S and S′ touch and $\Theta = 0$, we have

$$27\Delta\Delta'^2 + 4\Theta'^3 = 0;$$

therefore $(3\Delta'S + \Theta'S)(3\Delta'S - 2\Theta S)^2 = 0$.

But $3\Delta'S + \Theta'S' = 0$ breaks up into the tangent to S at P $\{x = a\cos\phi, y = b\sin\phi\}$, and $bx\sin\phi + ay\cos\phi + ab\sin\phi\cos\phi = 0$, which does not pass through P. It is therefore the required chord, the envelope being

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

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830. (M. Collins, B.A.)—From a point in a line of the third order that has a double point, only two tangents can be drawn to the curve; prove that the lines drawn from the double point to the points of contact, together with the two tangents at the double point, form a harmonic pencil.

Solution by Professor Nanson.

Any cubic circumscribing the triangle of reference is

$$ux^2 + vy^2 + wz^2 = kxyz,$$

where u, v, w are the tangents at the vertices. Take now as vertices the node and the points of contact of the two tangents. Then $w \equiv 0$ and the tangents at the node are of the form $ax^2 + by^2 = kxy$, whilst the condition that the two tangents meet on the curve is $k = 0$, which is also the condition that the tangents at the node form a harmonic pencil with the lines $xy = 0$ joining the node to the points of contact of the two tangents.

14906. (Professor Neuberg.)—A′, B′, C′ étant les pieds des hauteurs du triangle ABC, soient A_1, B_1, C_1 les symétriques des sommets A, B, C respectivement par rapport aux points A′, B′, C′. Calculer le rapport des aires des triangles $A_1B_1C_1$, ABC.

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Solution by Professor SANJÁNA, M.A.

If the perpendiculars meet in O, the area of the triangle OB_1C_1
$= \frac{1}{2}OB_1 . OC_1 \sin B_1OC_1 = \frac{1}{2}(OB' + BB')(OC' + CC') \sin A = \frac{1}{2}(BO + 2OB')$.
$(CO + 2OC') \sin A = 2R^2 (\cos B + 2 \cos C \cos A)(\cos C + 2 \cos B \cos A) \sin A$.
The areas of OC_1A_1, OA_1B_1 can be similarly expressed. Hence the area of $A_1B_1C_1$

$$= 2R^2 \{\Sigma \cos B \cos C \sin A + \Sigma (\cos^2 B + \cos^2 C) \sin 2A + 2 \cos A \cos B \cos C \Sigma \sin 2A\}$$
$$= 2R^2 \{\sin A \sin B \sin C + 2 \sin A \sin B \sin C + 8 \cos A \cos B \cos C \sin A \sin B \sin C\}$$
$$= \Delta \{3 + 8 \cos A \cos B \cos C\}.$$

Thus the required ratio is $3 + 8 \cos A \cos B \cos C : 1$.

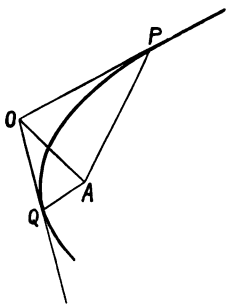
[N.B.—In the third line of the question, as printed originally, the order of the points is C′, A′, B′; this does not lead to any elegant result.]

14891. (A. M. NESBITT, M.A.)—ABC is a triangle, O any point. Tangents OP, OQ are drawn to the in-circle or one of the ex-circles. Show that OP touches the conic OABCQ, and OQ touches the conic OABCP.

Solutions (I.) *by* H. M. TAYLOR, M.A., F.R.S., H. W. CURJEL, M.A., *and others;* (II.) *by the* PROPOSER.

(I.) The theorem stated in the question for each of the four circles which touch the sides of a triangle is true for any conic which touches the sides of a triangle; and in this form it is a special case of the well-known theorem: "The six angular points of any two triangles which are described about a conic lie on a conic."

Let tangents at two consecutive points P, P′ meet the tangent at Q in the points O, O′; then, by the theorem quoted above, the points A, B, C, P, O, O′ lie on a conic, or, in other words, the conic ABCPO has the line OQ for a tangent.



(II.) Project B, C into the circular points at infinity; then the theorem at once takes this shape.

If two tangents OP, OQ be drawn to a parabola whose focus is A, OP will touch the circle OAQ, OQ will touch the circle OAP, which is at once obvious, since

$$\angle QOA = OPA,$$
$$\angle POA = OQA.$$

14971. (ROBERT W. D. CHRISTIE.) — Find factors of $N^4 + 1$ by an easy general solution where N^2 takes the form $(M^2 - 24)/10$. (See Quest. 13707.) *Ex.*: $14506^4 + 1 = 210469913$ and 210378169.

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Solution by the PROPOSER.

(1) When $n = 4m^2+1$ and $k^2 = 2m^2+1$, the solution of $M^2-2nN^2 = n^2-1$ is $M = 2kN$, $N = 2mk$.

(2) To solve $xz^2-(8r^2+2)x^2 = 16r^4+8r^2$.

Let $x = 2r$. Then $z^2 = 12r^2+4$, $k^2 = 3r^2+1$,

where $r = 1, 4, 15, 56, 209$, &c., by law $a_{n+1} = 4a_n - a_{n-1}$;
$k = 2, 7, 26, 97, 362$, &c., by same law;
$z = 4, 14, 52, 194, 724$, &c., by same law;
$x = 2, 8, 30, 112, 418$, &c., by same law;
$xz = 8, 112, 1560, 21728$, &c., by law $a_{n+1} = 14a_n - a_{n-1}$.

Therefore $(2rz)^2-(8r^2+2)(2r)^2 = 16r^4+8r^2$;

consequently, $(2r)^4+1 = (4r^3+4r^2+1+2rz)(4r^3+4r^2+1-2rz)$.

(3) The solution of $X^2-2nY^2 = (n^2-1)z^2$ immediately follows.

(4) Let $a = 2n$, $b = n^2-1$, $c = n^2+1$; then, if $y^2-ax^2 = b$, we have

$$(y^4-2by^2+c^2)/a^2 = x^4+1 = (x^2+n+y)(x^2+n-y).$$

In the example given, $y = 45872$, $x = 14506$, $n = 5$; thus the factors of $14506^4+1 = 210469913$ and 210378169. The solution of x^4+m^4

$$= (x^2+nm^2+my)(x^2+nm^2-my).$$

Here m only is arbitrary. Thus $n = 17$, $m = 3$ gives

$$36^4+3^4 = 2097 \times 801.$$

(5) The general solution of $(nrk)^4+1$, where $k^2 = \frac{1}{4}n^2r^2+1$, is

$$\tfrac{1}{4}n^4r^4+2n^2r^3+1 \pm n^3r^3+2nr,$$

or, taking $q = nr$, $(qk)^4+1 = \frac{1}{4}q^4+2q^2+1 \pm q^3+2q$.

Here n is perfectly arbitrary (and, if minus, we get the factors of nr^4-1, since $nr^4+1 = 2r^3+1 \pm 2r\sqrt{(\frac{1}{4}lr^2+1)}$, where $n+l = 4$).

Ex.: $408^4+1 = 27710263297 = 180913 \times 153169$.

This may, however, be more simply solved as follows:—

$$(2rk)^4+1 = 8r^4+8r^2+1 \pm 8r^3+4r \quad (r = 12,\ k = 17).$$

Here $k = 3, 17, 99$, &c.; $r = 2, 12, 70$, &c.;

according to law $k_{n+1} = 6k_n - k_{n-1}$, $r_{n+1} = 6r_n - r_{n-1}$.

The product nr may be varied to suit requirements, *e.g.*, $12^4+1 = 233 \times 89$ from $n = 2$, $r = 2$, $k = 3$, or $n = \frac{1}{2}$, $r = 8$, $k = 3$.

[For further developments, see PROPOSER's forthcoming work, *Chips from a Mathematical Workshop*, containing about 200 original theorems.]

14980. (R. KNOWLES, B.A.)—Prove that the sum to r terms of the series $1+2^3x+3^3x^2+\ldots+r^3x^{r-1}$ is

$$\{1+4x+x^2-(r+1)^3x^r+(3r^3+6r^2-4)x^{r+1} - (3r^3+3r^2-3r+1)x^{r+2}+r^3x^{r+3}\}/(1-x)^4.$$

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Solutions (I.) *by* ALETROP; (II.) *by* A. W. POOLE, B.A., Rev. T. WIGGINS, B.A., *and many others.*

(I.) On a $\quad S_{r+1}-S_r = (r+1)^3 x^r.$

Posons $\quad S_r = (r^3\alpha + 3r^2\beta + 3r\gamma + \delta)\, x^r + \epsilon$ ..... (1),

α, β, γ, δ, ϵ étant des indéterminées. Alors

$$S_{r+1} = \left\{(r+1)^3\alpha + 3(r+1)^2\beta + 3(r+1)\gamma + \delta\right\} x^{r+1} + \epsilon.$$

Des équations qui résultent de l'identification des deux valeurs de $S_{r+1}-S_r$, on tire

$$\alpha = \frac{1}{x-1}, \quad \beta = \frac{-1}{(x-1)^2}, \quad \gamma = \frac{x+1}{(x-1)^3}, \quad \delta = \frac{-1-4x-x^2}{(x-1)^4}.$$

Quant à ϵ, en admettant que $S_0 = 0$, il vient de (1) $\epsilon = -\delta$. Substitutions faites, on trouve bien pour S_r la valeur annoncée.

(II.)
$$\begin{aligned}
S &= 1 + 2^3x + 3^3x^2 + 4^3x^3 + \ldots + r^3x^{r-1};\\
-4xS &= -4x - 4.2^3x^2 - 4.3^3x^3 - \ldots - 4(r-1)^3x^{r-1} - 4r^3x^r;\\
6x^2S &= 6x^2 + 6.2^3x^3 + \ldots + 6(r-2)^3x^{r-1} + 6(r-1)^3x^r + 6r^3x^{r+1};\\
-4x^3S &= -4x^3 - \ldots - 4(r-3)^3x^{r-1} - 4(r-2)^3x^r - 4(r-1)^3x^{r+1} - 4r^3x^{r+2};\\
x^4S &= x^4 + \ldots + (r-4)^3x^{r-1} + (r-3)^3x^r + (r-2)^3x^{r+1} + (r-1)^3x^{r+2} + r^3x^{r+3}.
\end{aligned}$$

$$\therefore S(1-x)^4 = 1 + A_1'x + A_2'x^2 + A_3'x^3 + \sum_{s=4}^{s=r-1} A_s x^s + A_r'x^r + A_{r+1}'x^{r+1} + A_{r+2}'x^{r+2} + r^3x^{r+3}.$$

Now
$$\begin{aligned}
A_s &= (s+1)^3 - 4s^3 + 6(s-1)^3 - 4(s-2)^3 + (s-3)^3\\
&= s^3(1-4+6-4+1) + 3s^2(1-6+8-3) + 3s(1+6-16+9) + (1-6+32-27)\\
&= 0.
\end{aligned}$$

Also
$$\begin{aligned}
A_r &= (r+1)^3 + A_r'; && \text{therefore } A_r' = -(r+1)^3;\\
A_{r+1} &= (r+2)^3 - 4(r+1)^3 + A_{r+1}'; && \text{therefore } A_{r+1}' = 3r^3 + 6r^2 - 4;\\
A_{r+2}' &= (r-1)^3 - 4r^3 = -(3r^3 + 3r^2 - 3r + 1);\\
A_1' &= 8 - 4 = 4; \quad A_2' = 27 + 6 - 32 = 1; \quad A_3' = 64 - 108 + 48 - 4 = 0.
\end{aligned}$$

$$\therefore \quad S = \left\{1 + 4x + x^2 - (r+1)^3x^r + (3r^3 + 6r^2 - 4)x^{r+1} - (3r^3 + 3r^2 - 3r + 1)x^{r+2} + r^3x^{r+3}\right\}/(1-x)^4.$$

14896. (SALUTATION.)—In a circle (of radius = unity) is a chord of unknown length, and in the chord a point joined to the centre by a straight line which forms with the radius drawn to the nearest extremity of the chord an angle = α. The projection upon this radius of the aforesaid point is at a distance from the centre = $2\cos^2\alpha - \cos\alpha$. If $\cos\alpha = \frac{1}{4}\pi$, prove that the chord subtends an angle whose cosine = −·4183023.

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Solution by Rev. T. Wiggins, B.A.; *and* A. W. Poole, B.A.

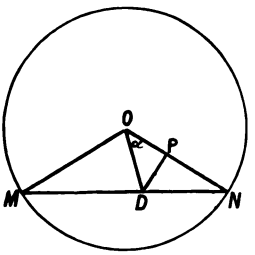
$NP = ON - OP = (1 - 2\cos^2\alpha + \cos\alpha)$;

$DN^2 = OD^2 + ON^2 - 2OD \cdot ON \cos\alpha$

$= OP^2 \sec^2\alpha + 1 - 2OP \sec\alpha \cos\alpha$

$= (2\cos\alpha - 1)^2 + 1 - 2(2\cos\alpha - 1)\cos\alpha$

$= 2(1 - \cos\alpha)$;



$\cos MON = \cos(180 - 2MNO) = -\cos 2MNO$

$= 1 - 2\cos^2 MNO = 1 - 2\dfrac{NP^2}{ND^2}$

$= 1 - \dfrac{2(1 - 2\cos^2\alpha + \cos\alpha)^2}{2(1 - \cos\alpha)}$

$= 1 - (1 + 2\cos\alpha)^2(1 - \frac{1}{4}\pi) = 1 - (1 + \frac{1}{2}\pi)^2(1 - \frac{1}{4}\pi)$

$= 1 - (1 + \frac{1}{2}\pi^2 + \pi)(1 - \frac{1}{4}\pi) = 1 - 1 - \frac{1}{2}\pi^2 - \pi + \frac{1}{8}\pi + \frac{1}{16}\pi^3 + \frac{1}{4}\pi^2$

$= \frac{1}{16}\pi^3 - \frac{3}{8}\pi = -\cdot 4183\ldots.$

14831. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{\sqrt{5}+3} + \frac{1}{2\sqrt{5}+3} + \frac{1}{5\sqrt{5}+3} + \frac{1}{13\sqrt{5}+3} + \ldots = \frac{1}{\sqrt{5}},$$

$$\frac{1}{\sqrt{6}+4} + \frac{1}{3\sqrt{6}+4} + \frac{1}{11\sqrt{6}+4} + \frac{1}{41\sqrt{6}+4} + \ldots = \frac{1}{\sqrt{12}},$$

$$\frac{1}{\sqrt{7}+5} + \frac{1}{4\sqrt{7}+5} + \frac{1}{19\sqrt{7}+5} + \frac{1}{91\sqrt{7}+5} + \ldots = \frac{1}{\sqrt{21}},$$

$$\frac{1}{\sqrt{2}+3} + \frac{1}{5\sqrt{2}+3} + \frac{1}{29\sqrt{2}+3} + \frac{1}{169\sqrt{2}+3} + \ldots = \frac{1}{2\sqrt{2}},$$

$$\frac{1}{\sqrt{3}+5} + \frac{1}{9\sqrt{3}+5} + \frac{1}{89\sqrt{3}+5} + \frac{1}{881\sqrt{3}+5} + \ldots = \frac{1}{2\sqrt{6}},$$

$$\frac{1}{2+7} + \frac{1}{26+7} + \frac{1}{362+7} + \frac{1}{5042+7} + \ldots = \frac{1}{4\sqrt{3}}.$$

Solution by H. W. Curjel, M.A.

Let $$\omega_n = \frac{x^{n-\frac{1}{2}} - x^{-n+\frac{1}{2}}}{x^{n-\frac{1}{2}} + x^{-n+\frac{1}{2}}}.$$

If the $(n+1)$-th term of a series

$$= \frac{1}{x^{2n+1} + x^{-2n-1} + x^2 + x^{-2}} = \frac{1}{(x^{n+\frac{1}{2}} + x^{-n-\frac{1}{2}})(x^{n-\frac{1}{2}} + x^{-n+\frac{1}{2}})} = \frac{(\omega_{n+2} - \omega_n)}{2(x^2 - x^{-2})},$$

the sum to $(n+1)$ terms $= \dfrac{1}{2(x^2 - x^{-2})}(\omega_{n+2} + \omega_{n+1} - \omega_1 - \omega_0)$,

and the sum to infinity $= \dfrac{1}{x^2 - x^{-2}}$, since $\omega_1 + \omega_0 = 0$ and $\omega_\infty = 1$, if $x > 1$.

In the given series let

$$x = \frac{\sqrt{5}+1}{2},\ \frac{\sqrt{3}+1}{\sqrt{2}},\ \frac{\sqrt{7}+\sqrt{3}}{2},\ \sqrt{2}+1,\ \sqrt{3}+\sqrt{2},\ 2+\sqrt{3}.$$

In the first three cases sum to infinity

$$= \frac{1}{x^2 - x^{-2}} = \frac{1}{\sqrt{5}},\ \frac{1}{\sqrt{(12)}},\ \frac{1}{\sqrt{(21)}};$$

in the last three cases sum to infinity

$$= \frac{2}{x^2 - x^{-2}} = \frac{1}{2\sqrt{2}},\ \frac{1}{2\sqrt{6}},\ \frac{1}{4\sqrt{2}}.$$

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Note on Quest. 14954, *evaluating the integral* $\int_0^\infty \frac{\sin(\frac{1}{2}r\pi + ax)}{x^{n-r}}\,dx$. *By* $\theta.\beta.$

It appears to have escaped notice entirely that the integral in question may have no meaning. In fact, the integral $\int_0^a \frac{\sin x}{x^m}\,dx$ $(a>0)$ has no meaning unless $m<2$; and $\int_0^a \frac{\cos x}{x^m}\,dx$ has none unless $m<1$; while $\int_a^\infty \frac{\sin x}{x^m}\,dx$ and $\int_a^\infty \frac{\cos x}{x^m}\,dx$ require $m>0$, in order that they may have a meaning.

Summing up, it appears that when r is *even* the integral

$$\int_0^\infty \frac{\sin(\frac{1}{2}r\pi + ax)}{x^{n-r}}\,dx$$

requires the condition $2>(n-r)>0$, and, when r is *odd*, the condition is $1>(n-r)>0$. Both of the cases are thus reduced to familiar results.

The first solution published affords an excellent illustration of the danger of differentiating under the integral sign without first examining whether such an operation is permissible. Consider, for instance, the integral

$$\phi(a) = \int_a^\infty \sin(ax) f(x)\,dx,$$

$f(x)$ being supposed such that $\phi(a)$ has a meaning. Then

$$\frac{1}{h}[\phi(a+h)-\phi(a)] = \int_a^\infty \frac{\sin(a+h)x - \sin(ax)}{h} f(x)\,dx$$
$$= \int_a^\infty \frac{\sin\frac{1}{2}hx}{\frac{1}{2}h} \cos(ax+\tfrac{1}{2}hx) f(x)\,dx,$$

and $\phi'(a)$ will be the limit of this expression (if such limit exists) as h tends to the limiting value zero. But, since the upper limit of the integral is infinite, we *cannot* say that, for *all* values of x within the range of integration $\mathrm{Lt}_{h=0} \frac{\sin\frac{1}{2}hx}{\frac{1}{2}h}\cos(ax+\frac{1}{2}hx) = x\cos(ax)$.

It may be possible, of course, if $f(x)$ has a suitable form, to avoid the difficulty; for instance, let $f(x)=e^{-x}$, and then we can write, if h is positive,

$$\frac{1}{h}[\phi(a+h)-\phi(a)] = \int_a^{1/\sqrt{h}} e^{-x}dx \frac{\sin\frac{1}{2}hx}{\frac{1}{2}h}\cos(ax+\tfrac{1}{2}hx)$$
$$+\int_{1/\sqrt{h}}^\infty e^{-x}dx \frac{\sin\frac{1}{2}hx}{\frac{1}{2}h}\cos(ax+\tfrac{1}{2}hx).$$

In the first of these we may put

$$\mathrm{Lt}_{h=0}\left\{(\sin\tfrac{1}{2}hx)/\tfrac{1}{2}h\right\}\cos(ax+\tfrac{1}{2}hx) = x\cos(ax),$$

while the second is, in absolute value, less than

$$\int_{1/\sqrt{h}}^\infty \frac{2}{h}e^{-x}dx = [2/(\sqrt{h})^2]e^{-1/\sqrt{h}},$$

which tends to zero with $\sqrt{h}$.

It has been assumed that h is positive; but a similar method can be

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applied when h is negative, and the same result is obtained. Thus we have

$$\phi'(a) = \int_a^\infty x \cos(ax)\, e^{-x} dx,$$

where

$$\phi(a) = \int_a^\infty \sin(ax)\, e^{-x} dx,$$

so that the ordinary rule is justified; but it has required a special proof. It is, unfortunately, impossible to refer to an *English* text-book in which such questions are adequately discussed; they are fully treated in good foreign books on the calculus, and in Mr. G. H. HARDY's recent papers.

14695. (R. KNOWLES.)—Prove that the sums to r terms of the series

(1) $1-(m+1)+(2m+1)-(3m+1)\ldots\pm(mr-m+1)$,

(2) $1-(m+1)^2+(2m+1)^2-(3m+1)^2\ldots\pm(mr-m+1)^2$,

(3) $1-(m+1)^3+(2m+1)^3-(3m+1)^3\ldots\pm(mr-m+1)^3$

are respectively

$-\frac{1}{2}rm$, $\frac{1}{2}rm(m-rm-2)$, and $-\frac{1}{4}rm\{r(2r-3)m^2+6(r-1)m+6\}$,

when r is even,

and $\frac{1}{2}(mr-m+2)$, $\frac{1}{2}rm\{(rm-m+2)-2(m-1)\}$,

and $\frac{1}{4}\{(r-1)^2(2r+1)m^3+6r(r-1)m^2+6(r-1)m+4\}$,

when r is an odd number.

Solutions (I.) *by* LIONEL E. REAY, B.A.; (II.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(I.) First, let r be even.

(1) $1-(m+1)+(2m+1)-\ldots-(mr-m+1)$
$= -m\{1-2+3-4+\ldots+(r-1)\}$
$= -m\left[1+(3-2)+(5-4)+\ldots+\{(r-1)-(r-2)\}\right] = -\frac{1}{2}mr.$

(2) $1-(m+1)^2+(2m+1)^2-\ldots-(mr-m+1)^2$
$= -2m\{1-2+3-4+\ldots+(r-1)\}-m^2\{1^2-2^2+3^2-\ldots+(r-1)^2\}$
$= -mr-m^2\left[1^2+2^2+3^2+\ldots+(r-1)^2-2\{2^2+4^2+\ldots+(r-2)^2\}\right]$
$= -mr-m^2\left[\frac{1}{6}\{r(r-1)(2r-1)\}-\frac{2}{6}\cdot\frac{1}{2}(r-2)\frac{1}{2}r(r-1)\right]$
$= -mr-m^2\frac{1}{6}\{r(r-1)\}(2r-1-2r+4) = -mr-m^2\frac{1}{2}\{r(r-1)\}$
$= \frac{1}{2}mr(m-mr-2).$

(3) $1-(m+1)^3+\ldots-(mr-m+1)^3$
$= -\frac{3}{2}mr-\frac{1}{2}\{3m^2r(r-1)\}-m^3\left[\{\frac{1}{2}(r-1)r\}^2-16\{\frac{1}{3}\cdot\frac{1}{2}(r-2)\frac{1}{2}r\}^2\right]$
$= -\frac{3}{2}mr-\frac{1}{2}\{3m^2r(r-1)\}-\frac{1}{4}r^2m^3\{(r-1)^2-(r-2)^2\}$
$= -\frac{1}{4}mr\{r(2r-3)m^2+6(r-1)m+6\}.$

Next, when r is odd. If we add the next term to each of the above results, we get the sum of $r+1$ terms. Then, writing $r-1$ for r, this gives the sum to r terms when r is odd.

(1) $-\frac{1}{2}rm+(mr+1) = \frac{1}{2}(mr+2)$; therefore sum is $\frac{1}{2}(mr-m+2)$.

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(2) $\frac{1}{2}rm(m-rm-2)+m^2r^2+2mr+1=\frac{1}{2}r^2m^2+\frac{1}{2}rm^2+rm+1$;

therefore sum is $\frac{1}{2}\{rm(rm-m+2)-2(m-1)\}$ on putting $r-1$ for r. (Note that the result given in the question contains a misprint.)

(3) $-\frac{1}{4}rm\{r(2r-3)m^2+6(r-1)m+6\}+(mr+1)^3$
$=\frac{1}{4}\{m^3r^2(2r+3)+6m^2r(r+1)+6mr+4\}$.

Writing $r-1$ for r, this becomes

$$\frac{1}{4}\{m^3(r-1)^2(2r+1)+6m^2r(r-1)+6m(r-1)+4\}.$$

(II.) Let the three sums be written S'_e, S''_e, S'''_e when r is *even* (ϵ), and S'_ω, S''_ω, S'''_ω when r is *odd* (ω); and let

$$s'_r=1+2+3+\ldots+r,\quad s''_r=1^2+2^2+3^2+\ldots+r^2,\quad s'''_r=1^3+2^3+3^3+\ldots+r^3.$$

When r is *even*, the sum of the units is *zero*; when r is *odd*, the sum of the units is 1.

$$\begin{aligned}
S'_e &= \{-1+2-3+4-\ldots+(r-2)-(r-1)\}m=\{-s'_{r-1}+2\cdot 2s'_{\frac{1}{2}(r-2)}\}m\\
&=\{-\tfrac{1}{2}[1+(r-1)](r-1)+4\cdot\tfrac{1}{2}[1+\tfrac{1}{2}(r-2)]\tfrac{1}{2}(r-2)\}m\\
&=\{-\tfrac{1}{2}r(r-1)+\tfrac{1}{2}r(r-2)\}m=-\tfrac{1}{2}rm.\\
S''_e &=\{-1^2+2^2-3^2+4^2-\ldots+(r-2)^2-(r-1)^2\}m^2+2S'_e\\
&=\{-s''_{r+1}+2\cdot 2^2\cdot s''_{\frac{1}{2}(r-2)}\}m^2+2S'_e\\
&=\{-\tfrac{1}{6}(r-1)r(2r-1)+8\cdot\tfrac{1}{6}\cdot\tfrac{1}{2}(r-2)\tfrac{1}{2}r(r-1)\}m^2+2S'_e\\
&=\tfrac{1}{6}r(r-1)\{-(2r-1)+2(r-2)\}m^2+2S'_e=\tfrac{1}{6}rm(m-rm-2).\\
S'''_e &=\{-1^3+2^3-3^3+4^3-\ldots+(r-2)^3-(r-1)^3\}m^3\\
&\qquad+3\{-s''_{r-1}+2\cdot 2^2\cdot s''_{\frac{1}{2}(r-2)}\}m^2+3S'_e\\
&=\{-s'''_{r-1}+2\cdot 2^3\cdot s'''_{\frac{1}{2}(r-2)}\}m^3-3\cdot\tfrac{1}{2}r(r-1)m^2-\tfrac{3}{2}rm\\
&=\{-\tfrac{1}{4}[(r-1)^2r^2]+16\cdot\tfrac{1}{4}\cdot\tfrac{1}{4}(r-2)^2\tfrac{1}{4}r^2\}m^3-\tfrac{3}{2}r(r-1)m^2-\tfrac{3}{2}rm\\
&=-\tfrac{1}{4}rm\{r(2r-3)m^2+6(r-2)m+6\}.
\end{aligned}$$

Similarly, the sums S'_ω, S''_ω, S'''_ω, where r is *odd*, may be found.

14988. (The late Professor CAYLEY, F.R.S.) — If $l^2+m^2+n^2+\ldots$, $l'^2+m'^2+n'^2+\ldots$ are each = 0, and s, s' denote positive integers, show that the integral

$$\int(lx+my+nz+\ldots)^s(l'x+m'y+n'z+\ldots)^{s'}\,dx\,dy\,dz\ldots$$

over the interior of the hyper-sphere $x^2+y^2+z^2+\ldots=1$ is = 0 or = a numerical multiple of $(ll'+mm'+nn'+\ldots)^s$, according as s' is not, or is, $=s$.

Solution by Professor NANSON.

It is obvious that $\int x^{s+s'}\,dx\,dy\ldots$ is zero or numerical according as $s+s'$ is odd or even. Hence $\int(lx+my+\ldots)^{s+s'}\,dx\,dy\ldots$ has the value zero or $k(l^2+m^2+\ldots)^{\frac{1}{2}(s+s')}$, according as $s+s'$ is odd or even. Hence, denoting $\int(lx+my+\ldots)^s(l'x+m'y+\ldots)^{s'}\,dx\,dy\ldots$ by I and operating with $\{l'(d/dl)+m'(d/dm)+\ldots\}^{s'}$, we see that, if $s+s'$ is odd, then I = 0; that, if $s+s'$ is even and $s'<s$, then every term in I has $l^2+m^2+\ldots$ for a

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factor; that, if $s = s'$, then I consists of a numerical multiple of $(ll' + mm' + \ldots)^s$, together with terms every one of which has $l^2 + m^2 + \ldots$ for a factor. Hence result stated follows.

15023. (R. Chartres.)—If C_r denote the combinations of n things r together, and D_r those of $(2n-1)$ things r together, sum the series

$$\frac{C_1C_1}{D_1} - \frac{C_2C_3}{D_3} + \frac{C_3C_5}{D_5} - \frac{C_4C_7}{D_7} + \ldots .$$

Solution by the Proposer.

Taking Legendre's n-th coefficient P_n (n even)

$$n\int_{-1}^{+1} x^n - \frac{n(n-1)}{2(2n-1)}\, x^{n-2} + \ldots = 0,$$

or

$$\frac{2n}{n+1} = \frac{C_1C_1}{D_1} - \frac{C_2C_3}{D_3} + \ldots .$$

14665. (H. MacColl, B.A.)—Each letter denoting a statement, and α' being the denial of α, show that $(ax = ay)(\alpha'x = \alpha'z)$ and $\alpha(x = y) + \alpha'(x = z)$ are equivalent when α is a *constant* (x, y, z being unrestricted), but not necessarily when α is a *variable*. [*Def.*: A statement is a *constant* when it is *either certain or impossible*; a *variable* when it is *neither certain nor impossible*.]

Solution by the Proposer.

Let ϕ denote $(ax = ay)(\alpha'x = \alpha'z)$, and let ψ denote $\alpha(x = y) + \alpha'(x = z)$. Employing my usual notation, we get by mere inspection

$$\alpha^\epsilon : \{\phi = (x = y)\}\ \{\psi = (x = y)\} : (\phi = \psi),$$
$$\alpha^\eta : \{\phi = (x = z)\}\ \{\psi = (x = z)\} : (\phi = \psi).$$

Hence, $\alpha^{\epsilon,\eta} : (\phi = \psi)$. That is, when α is a *certainty* (ϵ) or an *impossibility* (η), ϕ is equivalent to ψ.

But this equivalence does not necessarily hold when α is a *variable* (θ). It fails, for example, in the case $\alpha^\theta x^\eta y^\epsilon z^\eta$; for then (remembering that no variable or its denial can be impossible)

$$\phi = (\theta\epsilon = \theta\epsilon)(\theta'\epsilon = \theta'\eta) = \epsilon(\theta' = \eta) = \eta,$$
$$\psi = \theta(\epsilon = \epsilon) + \theta'(\epsilon = \eta) = \theta + \eta = \theta.$$

An *impossibility* is a statement whose chance of being true is *absolute zero*. The chance of a *variable* may be infinitesimal, but it is never zero; it is always > 0 and < 1. A *certainty* is a statement whose chance is absolute unity.

Note.—A statement is not necessarily a *variable* when we *do not know* whether it is true or false. From want of data we may be not only un-

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able to determine whether it is true or false, but even whether its chance is 0 or 1 or some fraction between the two. To assert that A is a *variable* is to assert that the chance of A is >0 and <1; and this is a conclusion which our data may not justify. On the other hand, *with proper data*, we may find answers to such questions as "What is the chance that the chance is 1?" "What is the chance that the chance is 0?" "What is the chance that the chance is $\frac{1}{3}$?" (See my article in *Mind*, January, 1900.)

In the preceding verification of the case of non-equivalence $\alpha^{\rho}x^{\sigma}y^{\tau}z^{\upsilon}$, instead of substituting the *generic* symbols θ, ϵ, ϵ, η respectively for α, x, y, z, we should, in strictness, have substituted the *determinate* symbols θ_1, ϵ_1, ϵ_2, η_1; but, as we have only *one* variable θ_1, and as any certainty ϵ_1 is equivalent as *factor* or *term* to any other certainty ϵ_2 (as any impossibility η_1 to any impossibility η_2) the numerical subscripta may be left understood. When, however, two or more *variables* θ_1, θ_2, &c., enter into our argument, the suppression of the numerical subscripta in their case might lead to error; for θ_1 is not necessarily equivalent as factor or term to θ_2. Consider, for example, the case $\alpha^{\rho}x^{\sigma}y^{\tau}z^{\upsilon}$. Here, putting $\alpha = \theta_1$, $x = \rho_2$, $y = \eta$, $z = \epsilon$, we get

$$\phi = (\theta_1\theta_2 = \theta_1\eta)(\theta_1'\theta_2 = \theta_1'\epsilon) = (\theta_1\theta_2 = \eta)(\theta_1'\theta_2 = \theta_1') = (\theta_1 : \theta_2')(\theta_1' : \theta_2) = (\theta_2 = \theta_1'),$$

$$\psi = \theta_1(\theta_2 = \eta) + \theta_1'(\theta_2 = \epsilon) = \theta_1\eta + \theta_1'\eta = \eta.$$

In this case, therefore, when $\theta_2 = \theta_1'$ (that is, when x is the denial of α) ϕ is true and ψ false.

This Question was suggested by the results which I obtained in my solution of Professor SAVAGE's Quest. 14394. Putting x for AB, y for $x_{3'.1}$, z for $x_{3'.1} + x_{0'.4}$, α for n^{ρ}, and consequently α' for n^{0}, my answer to 14394 will then be the statement which I here express by ψ. But, since in 14394 α is either ϵ (included in our data), or η (in contradiction to our data), we have $\alpha^{\rho,\tau}$; so that, in this case, the answer may also be expressed in the form ϕ.

9848. (Professor HANUMANTA RAU.)—Solve the equation

$$x^3 - 3mx^2 + 3x - m = 0.$$

Solutions (I.) *by* Professor S. SIRCOM; (II.) *by* Rev. T. WIGGINS, B.A., F. W. REEVES, B.A.; *and others.*

(I.) If $m < 1$, $x = \tanh(2r\iota\frac{1}{3}\pi + \alpha)$, $r = 0, 1, 2$, where $m = \tanh 3\alpha$.
If $m > 1$, $x = \coth(2r\iota\frac{1}{3}\pi + \alpha)$, $r = 0, 1, 2$, where $m = \coth 3\alpha$.

(II.) For x substitute $(a + m)$, and the equation becomes

$$a^3 + 3a(1 - m^2) + 2m(1 - m^2) = 0.$$

For a substitute $(y + z)$, and the new equation becomes

$$y^3 + z^3 + 3yz(y + z) + 3(1 - m^2)(y + z) + 2(1 - m^2) = 0.$$

This is satisfied if $\qquad y^3 + z^3 = -2(1 - m^2)$,

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and $yz = -(1-m^2)$ or $y^3z^3 = -(1-m^2)^3$.

Hence y^3 and z^3 are the roots of $t^2 + 2m(1-m^2)t - (1-m^2)^3 = 0$; therefore (by solving) $y^3 = (m^2-1)(m+1)$, $z^3 = (m^2-1)(m-1)$; whence

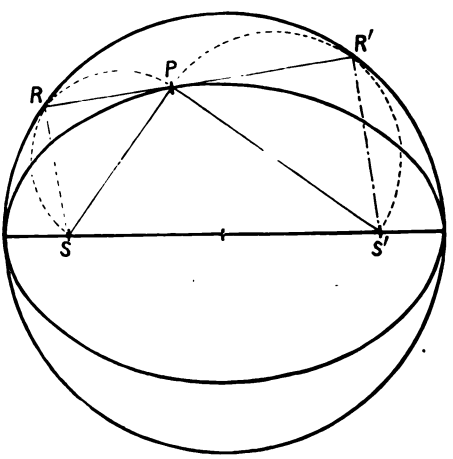
$$x = a + m = y + z + m = \sqrt[3]{(m^2-1)}\,[\sqrt[3]{(m+1)} + \sqrt[3]{(m-1)}] + m,$$

which is the real root.

14978. (Rev. T. Roach, M.A.)—Circles are described on SP, S'P in an ellipse as diameters and meet the auxiliary circle in R, R'; show that RR' is the tangent at P.

Solution by M. R. Peiris *and* Lionel E. Reay, B.A.

Since circle SPR is described on SP as diameter, this circle touches the auxiliary circle at R; therefore SRP is a right angle and RP touches



the ellipse. Similarly, R'P touches the ellipse, therefore RPR' is the tangent at P.

14761. (A. F. van der Heyden, B.A.)—Let I, I_1, I_2, I_3 denote the in- and ex-centres of a triangle. Given the circum-centres of the triangles II_2I_3, II_3I_1, and of the original triangle, construct the triangle.

Solutions (I.) *by* H. W. Curjel, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) I is the orthocentre of $I_1I_2I_3$, and the circum-circle (centre O) of triangle ABC is the nine-point-circle of triangle $I_1I_2I_3$. Hence the circum-circles of II_1I_3, II_1I_2, II_2I_3, $I_1I_2I_3$ are all equal. Therefore the middle point F of the line joining the centres L, M of circles II_2I_3, II_1I_3 (*i.e.*, middle point of II_3) lies on circle ABC. Therefore the circle ABC can

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be described, and line II_3 drawn; where it cuts the circle ABC again is point C, and I_1I_2 passes through C and is at right angles to II_3. Also I_2OM, I_1OL are straight lines; hence I_2, I_1 are determined, and hence also I, I_3 as intersections of circles with centres L, M and radii LI_2, MI_1. And the intersections of II_1, I_1I_3 with I_2I_3, I_2I are the points A and B.

(II.) Bisecting II_1, II_2, II_3 at right angles by the lines O_2LO_3, O_3MO_1, O_1NO_2, we get O_1, O_2, O_3 as the circum-centres of II_2I_3, II_3I_1, II_1I_2, respectively. Now

$$\begin{aligned}\angle O_1I_3I_2 &= \tfrac{1}{2}(\pi - I_2O_1I_3)\\ &= \tfrac{1}{2}\{\pi - (2\pi - 2I_2II_3)\}\\ &= I_2II_3 - \tfrac{1}{2}\pi\\ &= BIC - \tfrac{1}{2}\pi = \tfrac{1}{2}A,\end{aligned}$$

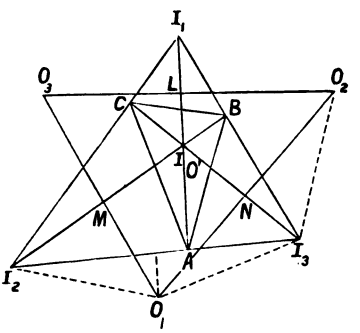
and $\angle II_3I_2 = \tfrac{1}{2}B$;

hence $\angle NI_3O_1 = \tfrac{1}{2}(B + A)$.

So also $\angle O_2I_3I_1 = \tfrac{1}{2}B$

and $\angle II_3I_1 = \tfrac{1}{2}A$;

hence $\angle NI_3O_2 = \tfrac{1}{2}(B + A)$.



Therefore N is the mid-point of O_1O_2. So L and M are the mid-points of O_2O_3, O_3O_1. Thus the circle ABC is the nine-point-circle of $O_1O_2O_3$, as it goes through L, M, N. Hence, of the triangle $O_1O_2O_3$, two angular points O_1 and O_2 are given, and the nine-point-centre O is also given; this triangle therefore can be easily constructed. With centres O_1, O_2, O_3 and radii each equal 2ON, strike arcs meeting two and two in I_1, I_2, I_3; the pedal triangle of $I_1I_2I_3$ is the original triangle.

15006. (Professor Ignacio Beyens, Lt.-Col. du Génie.)—Soit SABC un angle trièdre trirectangle; SD une droite intérieure que l'on projette sur les faces SAB, SAC, SBC, suivant SC′, SB′, SA′; démontrer que $A'SB' + A'SC' + B'SC' = 180°$, et déduire de ce théorème la propriété correspondante du triangle sphérique (démonstration géométrique).

Solution by Aletrop.

Considérons le parallélipipède dont l'un des angles trièdres est SABC et SD une des diagonales. Menons A′B′, B′C′, C′A′. Par avoir un côté commun et les deux autres côtés égales comme diagonales de rectangles égales, on a $\triangle SA'B' = SB'C'$, d'où $\angle A'SB' = SB'C'$;

$\triangle SA'C' = SB'C'$, d'où $\angle A'SC' = SC'B'$;

par suite $A'SB' + A'SC' + B'SC' = SB'C' + SC'B' + B'SC' = 180°$.

De là on déduit aisément: Le périmètre du triangle pédal d'un point pris dans l'intérieur d'un triangle sphérique trirectangle (de référence) est égal à une demi-circonférence de cercle maxime.

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14970. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Show how to find a set of three or more numbers q_r such that $(x^{q_0}-1)$, $(x^{q_1}-1)$, $(x^{q_2}-1)$, &c., are divisible by q_1, q_2, q_3, &c., *respectively* for several assigned values of x ($= a, b, c,$ &c.) *at once*. Give examples when x has *each* of the values 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30 for the same set of exponents q_0, q_1, q_2, &c., with the respective divisors q_1, q_2, q_3, &c.

Solution by the PROPOSER.

Let q_1, q_2, q_3, &c. be a succession of *primes* such that

$$q_1 = 2q_0+1, \quad q_2 = 2q_1+1, \quad q_3 = 2q_2+1, \quad \&c. \ldots\ldots (1);$$

whence $$q_0 = \tfrac{1}{2}(q_1-1), \quad q_1 = \tfrac{1}{2}(q_2-1), \quad \&c. \ldots\ldots (1A).$$

Then the question requires

$$x^{q_0} \equiv 1 \pmod{q_1}, \quad x^{q_1} \equiv 1 \pmod{q_2}, \quad \&c. \ldots\ldots (2);$$

so that x should be a 2-ic residue of q_1, q_2, q_3, &c. ..... (2A).

This is satisfied if $q^r = 4m_r.x-1$ for *each* $r = 1, 2, 3,$ &c. ..... (3).

Now, taking $$q_1 = 4m.x-1 \ldots\ldots (4)$$

gives, by (1), $q_1 = 4(2m).x-1$, $q_2 = 4(4m).x-1$, &c..... (4A),

which are all of the required form (3). If several of these (q^r) in succession be *prime*, they would satisfy (2), and therefore solve the problem for any particular base x.

Next, giving x several values $x = a, b, c,$ &c., and taking

$$X = \text{L.C.M. of } a, b, c, \&c. \ldots\ldots (5),$$

and, taking $q_1 = 4mX-1$, giving, by (1),

$$q_2 = 4(2m)X-1, \quad q_3 = 4(4m)X-1, \quad \&c. \ldots\ldots (5A),$$

these q's are *all* of the required form (3) for *each* of $a, b, c,$ &c.; and, if several of them in succession were *prime*, they would also satisfy (2) for *each* base $a, b, c,$ &c.; and would also satisfy (2) for any base $x = a^{\alpha}.b^{\beta}.c^{\gamma}$, &c.

This completes the general solution.

Ex.—Taking $x = 2, 3, 5$ gives $X = 30$, $y_1 = 120m-1$; q_2, q_3, &c., are given by (1), and $q_4 = \frac{1}{2}(q_1-1)$.

On actual trial with various values of m, the following successions of *primes* q_1, q_2, q_3, &c. result, and therefore solve the problem for *all* the bases named:—

m;	q_0;	q_1;	q_2;	q_3;	q_4.
3;	179;	359;	719;	1439;	2879.
41;	1229;	2459;	4919;	9839;	... .
45;	2699;	5399;	10799;	21599;	... .

14959. (R. KNOWLES, B.A.)—P, Q are the ends of a diameter of a conic, centre C; R, R′ are the points through which pass all the chords which subtend a right angle at P, Q respectively. Prove that R, R′, C are collinear.

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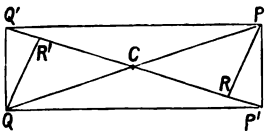
Solutions (I.) *by* JAMES BLAIKIE, M.A.; (II.) *by* R. TUCKER, M.A.; (III.) *by* LIONEL E. REAY, B.A., *and others*.

(I.) Let a circle described on PQ as diameter meet the conic in P′, Q′; then the normals at P and Q will meet the diameter P′Q′ in R, R′, the points through which pass all the chords which subtend a right angle at P, Q respectively. Thus R, C, R′ are collinear.



(II.) R, R′ are the FRÉGIER points for P, Q, and their coordinates (see LONEY, *Conics*, ch. xii., EX. XXXV., 29, or other text-books) are $\left(\frac{a^3e^2\cos\alpha}{a^2+b^2}, \frac{-a^2be^2\sin\alpha}{a^2+b^2}\right)$ and $\left(\frac{-a^3e^2\cos\alpha}{a^2+b^2}, \frac{a^2be^2\sin\alpha}{a^2+b^2}\right)$; whence, &c.

(III.) R is the FRÉGIER point of P and is on the normal at P. Similarly, R′ is on the normal at Q. By the symmetry of the figure PR = QR′; therefore, at once, R, C, and R′ are collinear.

[Other solutions, too numerous to publish, have been received.—EDITOR.]

14725. (ROBERT W. D. CHRISTIE.)—Prove the following theorems on circulators. P = any prime having an *even* period; H_1 = the number formed by the figures of the first half; H_2 = the number formed by the figures of the second half. Then

$$P-1 = (H_2+1)/(H_1+1), \quad H_2(P-1) = (P-2)\,10^{\frac{1}{2}p} + H_1 \ldots (1, 2),$$

$$(H_2+1)/(H_1+1) = (10^{\frac{1}{2}p}-H_1)/(10^{\frac{1}{2}p}-H_2), \quad 10^{\frac{1}{2}p}-H_1 = 0 \bmod P \ldots (3, 4),$$

$$H_1+H_2+2 = 0 \bmod P = 10^{\frac{1}{2}p}+1 \ldots\ldots (5),$$

$$(H_1+H_2+2)/(H_2-H_1) = P/(P-2) \ldots\ldots (6),$$

$$P(H_1+1) = 10^{\frac{1}{2}p}+1 \ldots\ldots (7),$$

and various others by combination.

Example of (1)— $P = 7$, $\frac{1}{7} = 142857$;

therefore $7-1 = (857+1)/(142+1) = 6$.

Solution by ALETHOP; H. W. CURJEL, M.A.; *and* A. M. NESBITT, M.A.

Soit p le nombre pair de chiffres de la période de 1/P. On a

$$1/P = (10^{\frac{1}{2}p}H_1 + H_2)/(10^p-1) \ldots\ldots (a).$$

D'ailleurs, on sait bien que, p étant *pair*,

$$H_1+H_2 = 999\ldots(\text{jusqu'à } \tfrac{1}{2}p \text{ chiffres}) = 10^{\frac{1}{2}p}-1 \ldots\ldots (b).$$

De (a) et (b) on tire

$$H_1 = 1/P\,(10^{\frac{1}{2}p}+1)-1 \ldots\ldots (c),$$

$$H_2 = 10^{\frac{1}{2}p}-1/P\,(10^{\frac{1}{2}p}+1) \ldots\ldots (d).$$

Maintenant, il est facile de vérifier les égalités proposées. Tout d'abord

$$H_1+1 = 1/P\,(10^{\frac{1}{2}p}+1), \quad H_2+1 = (P-1)/P\,(10^{\frac{1}{2}p}+1) \ldots\ldots (e, f),$$

d'où, par division, (1). Multipliant (d) par P−1 et en egard de (c), il

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vient (2). En divisant les valeurs de H_2+1 et H_1+1 tirés de (*b*), on a (3). De cette dernière et de (1),

$$10^{ip}-H_1=(10^{ip}-H_2)(P-1)=0 \bmod (P-1) \quad \ldots\ldots\ldots\ldots(4).$$

De (1), après avoir ajouté 1 aux deux membres,

$$H_1+H_2+2=(H_1+1)\,P=0 \bmod P \quad \ldots\ldots\ldots\ldots\ldots(5).$$

L'addition de (*e*) et (*f*) donne $H_1+H_2+2=10^{ip}+1$ .....(5).
Cette-ci divisée par la différence entre (*f*) et (*e*) fournit (6). Enfin, (7) est (*e*) mise sous forme entière.

14967. (R. Tucker, M.A.)—The tangent at P (ap^2, ap^3) to the curve $ay^2=x^3$ (A) meets the curve again at Q; prove that (1) the normals at P, Q meet, in R, on the curve $6561y^4+2430axy^2+32a^2y^2=ax\,(75x+8a)^2$; (2) the evolute to (A) is $9y^2\,(729y^2+1728ax+32a^2)+512ax\,(6x+a)^2=0$. (Gregory's *Exercises*, p. 195, gives a surd result.) If $9p^2=2$, then PR is also a tangent. Find the locus of the intersection of orthogonal tangents, and also of the circum-centre of POQ (O the cusp), and show that the mid-point of PQ lies on another semicubical parabola.

Solution by the Proposer *and* Lionel E. Reay, B.A.

1. Tangent at P is $2y-3px+ap^3=0$ .....(i.).

This meets the curve in $p'=-\frac{1}{2}p$. The normals at P, Q are

$$\left.\begin{aligned}3py+2x&=a\,(3p^4+2p^2)\\-24py+32x&=a\,(3p^4+8p^2)\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii.)},$$

After some reduction we have to eliminate p between

$$729y^2p^2-(1620xy+144ay)\,p+900x^2+96ax=0$$

and $$2ap^2+9yp-10x=0;$$

whence $81y^2(16a+225x)(729y^2+360ax+32a^2)=x\,(3645y^2+900ax+96a^2)^2$.
When cleared we get the result.

2. To obtain the evolute we eliminate p between (ii.) and $3y=12ap^3+4ap$. Reducing, we get the equations

$$27yp^2+4\,(6x+a)\,p-3y=0, \quad 4ap^2-9py-8x=0.$$

Ultimately we have

$$144y^2\,(18x-a)^2=(243y^2+96ax+16a^2)(27y^2+192x^2+32ax).$$

This reduces to the result in question.

3. If (ii.) is a tangent, we have $3pp'^3+2p'^2=3p^4+2p^2$. This gives for contact $(3p^2+2)(9p^2-2)=0$; hence $9p^2=2$.

4. For locus of intersection of orthogonal tangents, we have (i.) and $2y-3\lambda x=-a\lambda^3$, where $9p\lambda+4=0$; *i.e.*, multiplying,

$$4y^2-6xy\,(p+\lambda)-4x^2=-\tfrac{64}{729}a^2, \quad \text{also} \quad 3x=a\,(p^2+\lambda^2+p\lambda);$$

whence $$(27x-4a)\,x^2y^2=4a\,(y^2-x^2+\tfrac{16}{729}a^2)^2.$$
This reduces to $(729)^2\,(27x-4a)\,x^2y^2=4a\,[729\,(y^2-x^2)+16a^2]^2$.

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5. If equation to circle OPQ is $x^2+y^2=2ax+2\beta y$, we get

$$ap^2+6\beta p-10a=0, \quad 9\beta^2p^2-(30a+2a)\,\beta p+2a\alpha+25a^2=0;$$

whence locus is $a\alpha(25\alpha+2a)^2=4\beta^2(4\alpha^2+45a\alpha+108\beta^2)$.

6. The locus is readily got to be $125y^2=98x^3$.

11264. (Professor Zerr.)—Show that the intrinsic equation to the evolute of the curve $r^m=a^m\cos m\theta$ is

$$S'=\frac{a}{m+1}\left\{\left(\cos\frac{m}{m+1}\phi\right)^{(1-m)/m}-1\right\}, \text{ where } \phi \text{ has its usual meaning.}$$

Solution by the Proposer.

We easily get that $dr/d\theta=r\tan(\theta-\phi)$. From the equation to the curve $dr/d\theta=-a(\cos m\theta)^{(1-m)/m}\sin m\theta=-r\tan m\theta$; therefore

$$\tan(\theta-\phi)=-\tan m\theta, \quad \phi-\theta=m\theta, \quad \text{or} \quad \theta=1/(m+1)\,\phi \ldots\ldots(1).$$

$S'=\rho-C$ is the equation to the evolute $ds/d\theta=a(\cos m\theta)^{(1-m)/m}$, but $d\theta=1/(m+1)\,d\phi$; therefore $ds/d\phi=a/(m+1)[\cos m/(m+1)\,\phi]^{(1-m)/m}=\rho$; therefore $S'=a/(m+1)[\cos m/(m+1)\,\phi]^{(1-m)/m}-C$, when $S'=0$, $\phi=0$, and $C=a/(m+1)$; therefore $S'=a/(m+1)\left\{[\cos m/(m+1)\,\phi]^{(1-m)/m}-1\right\}$.

14843. (Professor Umes Chandra Ghosh.)—If K be the symmedian point of a triangle ABC, and KA, KB, KC produced cut the circum-circle of the triangle ABC at A_1, B_1, C_1 respectively, and if π_1, π_2, π_3 be the lengths of the perpendiculars from K on B_1C_1, C_1A_1, A_1B_1, prove that $1/a^2+1/b^2+1/c^2=\frac{1}{4}(\tan^2\omega)(1/\pi_1^2+1/\pi_2^2+1/\pi_3^2)$, where ω is the Brocard angle of the triangle ABC.

Solution by Professor Sanjána, M.A.

The line BB_1 is $\alpha/a=\gamma/c$, and the circum-circle is $a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$; hence at B_1 we have $\alpha/a=-2\beta/b=\gamma/c$; so also at C_1, $\alpha/a=\beta/b=-2\gamma/c$; thus the line B_1C_1 is $-bc\alpha+2ca\beta+2ab\gamma=0$. The perpendicular on this from K (ka, kb, kc) being π_1, we have

$$\frac{1}{\pi_1^2}=\frac{b^2c^2+4c^2a^2+4a^2b^2-8a^2bc\cos A+4b^2ca\cos B+4c^2ab\cos C}{(-kabc+2kbca+2kcab)^2}.$$

Similarly π_2 and π_3 may be obtained, and we thus get

$$\Sigma\frac{1}{\pi^2}=\frac{9b^2c^2+9c^2a^2+9a^2b^2}{9a^2b^2c^2k^2}=\frac{1}{k^2}\Sigma\frac{1}{a^2}.$$

Hence

$$\Sigma\frac{1}{a^2}=k^2\Sigma\frac{1}{\pi^2}=\tfrac{1}{4}\tan^2\omega\,\Sigma\frac{1}{\pi^2}.$$

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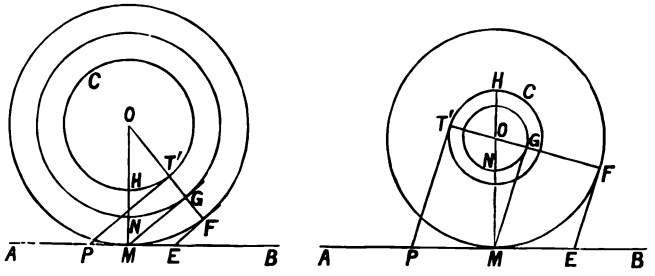
14803. (H. D. Drury, M.A.)—C is a given circle, APB a given straight line: to find a point P in AB so that, when tangents PT, PT′ are drawn to C, $\angle APT = 2BPT'$.

Solutions (I.) *by* R. F. Davis, M.A.; (II.) *by* H. W. Curjel, M.A.

(I.) Let O be the centre of C, and O′ the image of O in AB. Describe the circle whose centre is O′ which touches AB. Any common tangent to this circle and the circle C will intersect AB in P, the required point. (Four solutions.)

(II.) Let the perpendicular to AB from the centre O of the circle C meet the circle in H and AB in M. Bisect MH in N and with centre O and radius ON describe a circle, and to this circle draw a tangent MG from M. Let OG meet the circle C in T′, and circle with centre O and



radius OM in F, T′ and F being equidistant from G. Draw tangents T′P and FE to these circles at T′ and F. Then PM = ME. Hence $\angle FOM = 2\angle MOP$, and therefore, if the other tangent PT from P to the circle C be drawn, $\angle APT = 2\angle BPT'$.

14739. (J. A. Third, M.A., D.Sc.)—M and N are points on the sides CA, AB of a triangle, such that CA . CM = BA . BN. M′ and N′ are the images of M and N with respect to BC. Through M′ and N′ parallels are drawn to CA, AB respectively, intersecting in P. AP meets BC in D. The connector of A with the circum-centre of ABC meets BC in E. Show that BD = CE.

Solutions (I.) *by* H. W. Curjel, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) Let PM′, PN′ cut BC in C′ and B′, and let O be the circum-centre of $\triangle ABC$. Then

$$\begin{aligned} CD : DB &= \triangle PAC : PAB = \triangle C'CA : B'BA \\ &= C'M \cdot CA \sin AMC' : B'N \cdot AB \sin ANB' \\ &= CM \cdot CA \sin 2C : BN \cdot BA \sin 2B \\ &= \sin 2C : \sin 2B = \triangle OAB : OAC = EB : EC; \end{aligned}$$

therefore $BD = CE$.

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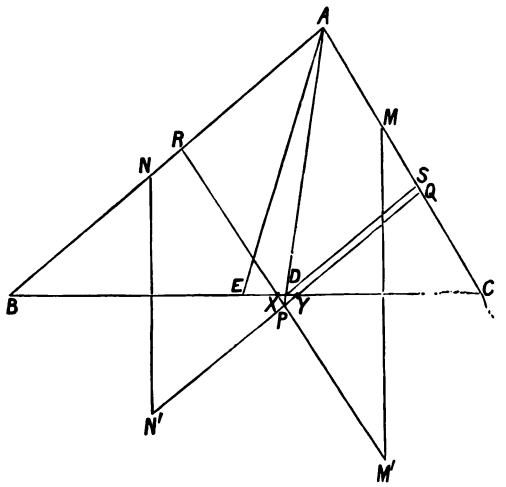
(II.) The proposition can be easily proved by using barycentric co-ordinates. The following is an elementary proof.

Let N'P, M'P meet CA, AB in Q, R; let DS be parallel to AR. Then

$$AR + AQ = DS + AS = \frac{DC \sin C}{\sin A} + \left(b - \frac{DC \sin B}{\sin A}\right) = \frac{DC \sin C}{(a - DC) \sin B}.$$



Again, $$\frac{AR}{AB} = \frac{CX}{CB} = \frac{2CM \cos C}{CB} = \frac{2CM \,.\, CA \cos C}{ab},$$

and $$\frac{AQ}{AC} = \frac{2BN \,.\, BA \cos B}{ac};$$

hence $AR + AQ = c^2 \cos C + b^2 \cos B$. Equating these two values, we get

$$DC = \frac{ac \cos C}{b \cos B + c \cos C}.$$

Now $$BE = \frac{c \cos C}{\sin(\frac{1}{2}\pi - B + C)} = \frac{c \cos C}{\cos(B - C)} = \frac{2c \sin A \cos C}{\sin 2B + \sin 2C}$$

$$= \frac{ac \cos C}{b \cos B + c \cos C}. \quad \text{Hence } BD = CE.$$

14059. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{3^2+1} + \frac{1}{7^2-1} + \frac{1}{17^2+1} + \frac{1}{41^2-1} + \ldots = \tfrac{1}{8},$$

$$\frac{1}{1^2+9} - \frac{1}{7^2+9} + \frac{1}{41^2+9} - \frac{1}{239^2+9} + \ldots = \tfrac{1}{12},$$

$$\frac{1}{3^2-1} + \frac{1}{11^2-1} + \frac{1}{41^2-1} + \frac{1}{153^2-1} + \ldots = 1 - \tfrac{1}{2}\sqrt{3}.$$

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Solution by Professor SANJÁNA, M.A.

In the recurring series 1, 3, 7, 17, 41, 99, 239, ..., it is seen that

$$3^2+1 = 2.1.5, \quad 17^2+1 = 2.5.29, \quad 99^2+1 = 2.29.169, \ \ldots,$$

where 1, 5, 29, 169, ... is a recurring series whose scale is $1-6+1$. Hence

$$\frac{1}{3^2+1}+\frac{1}{17^2+1}+\ldots = \tfrac{1}{2}\left\{\frac{1}{1.5}+\frac{1}{5.29}+\frac{1}{29.169}\ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{1.5-}\ \frac{1^2.5^2}{1.5+5.29-}\ \frac{5^2.29^2}{5.29+29.169-}\ \ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{5-}\ \frac{5^2}{5.6.5-}\ \frac{5^2.29^2}{29.6.29-}\ \ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{5-}\ \frac{1}{6-}\ \frac{1}{6-}\ \ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{5-(3-2\sqrt{2})}\right\} = \frac{1}{4(\sqrt{2}+1)} = \frac{\sqrt{2}-1}{4}.$$

Also $7^2-1 = 8.1.6, \quad 41^2-1 = 8.6.35, \quad 239^2-1 = 8.35.204, \ \ldots,$

where 1, 6, 35, 204, ... is a recurring series whose scale is $1-6+1$. Hence, as before,

$$\frac{1}{7^2-1}+\frac{1}{41^2-1}+\ldots = \tfrac{1}{8}\left\{\frac{1}{1.6}+\frac{1}{6.35}+\frac{1}{35.204}+\ldots\right\}$$

$$= \tfrac{1}{8}\left\{\frac{1}{6-}\ \frac{1}{6-}\ \frac{1}{6-}\ \ldots\right\} = \tfrac{1}{8}(3-2\sqrt{2}) = \frac{1\frac{1}{2}-\sqrt{2}}{4}$$

Thus the first series $= \frac{1}{2}/4 = \frac{1}{8}$. Similarly,

$$1^2+9 = 2.1.5, \quad 41^2+9 = 2.5.169, \quad 1393^2+9 = 2.169.5741, \ \ldots;$$

so that

$$\Sigma\frac{1}{1^2+9} = \tfrac{1}{2}\left\{\frac{1}{1.5}+\frac{1}{5.169}+\frac{1}{169.5741}+\ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{5-}\ \frac{1}{34-}\ \frac{1}{34-}\ \ldots\right\} \quad \text{(for the scale of relation is } 1-34+1)$$

$$= \tfrac{1}{2}\left\{\frac{1}{5-(17-12\sqrt{2})}\right\} = \frac{\sqrt{2}+1}{24}.$$

And $$7^2+9 = 2.1.29, \quad 239^2+9 = 2.29.985,$$
$$8119^2+9 = 2.985.33461, \ \ldots;$$

so that $$\Sigma\frac{1}{7^2+9} = \tfrac{1}{2}\left\{\frac{1}{1.29}+\frac{1}{29.985}+\frac{1}{985.33461}+\ldots\right\}$$

$$= \tfrac{1}{2}\left\{\frac{1}{29-(17-12\sqrt{2})}\right\} = \frac{\sqrt{2}-1}{24}.$$

Thus the second series $= \frac{1}{12}$. The third is easily seen to be

$$= \tfrac{1}{2}\left\{\frac{1}{1.4}+\frac{1}{4.15}+\frac{1}{15.56}+\ldots\right\} = \tfrac{1}{2}\left\{\frac{1}{4-}\ \frac{1}{4-}\ \frac{1}{4-}\ \ldots\right\}$$
$$= \tfrac{1}{2}(2-\sqrt{3}) = 1-\tfrac{1}{2}\sqrt{3}.$$

14921. (Rev. T. WIGGINS, B.A.)—At right angles to a passage $5a$ feet wide is another passage $a\sqrt{3}$ feet wide. Find the greatest length of a rectangular table a feet wide which can be wheeled from one passage into the other.

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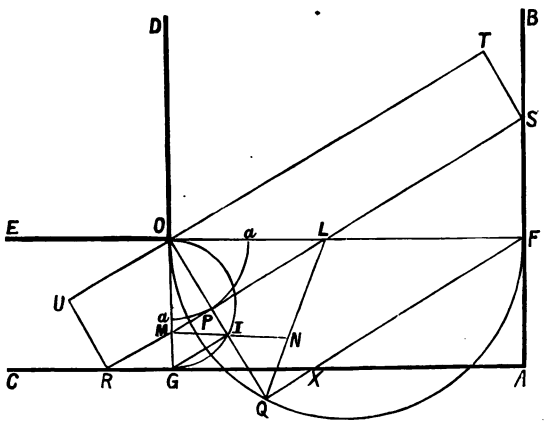
Solution by D. BIDDLE.

Let O be the obstructive corner at the junction of the two passages. Produce EO, DO to F, G respectively, and with centre O and radius = a describe the quadrant of a circle between the two produced lines. Then



the length required is that of the minimum intercept between AB and AC that is also tangential to the aforesaid quadrant.

Let P be the point of contact of this intercept with the quadrant, and let $\angle FOP = \theta$; then

$$l = RS = a\{(\sqrt{3} - \sin\theta)\sec\theta + (5 - \cos\theta)\operatorname{cosec}\theta\}$$
$$= a(\sqrt{3}\sec\theta - \tan\theta + 5\operatorname{cosec}\theta - \cot\theta).$$

Taking the differential coefficient of this value in respect of θ it is found to be zero when $\sqrt{3}\sin\theta - 1 : 5\cos\theta - 1 = \cos^2\theta : \sin^2\theta = \cot\theta : \tan\theta$. This holds good when $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{1}{2}\sqrt{3}$. Thus $\sec\theta = 2$, $\tan\theta = \sqrt{3}$, $\operatorname{cosec}\theta = 2/\sqrt{3}$, $\cot\theta = 1/\sqrt{3}$, and $l = 4a\sqrt{3}$.

15014. (Professor NANSON.)—A conic touches BC, CA, AB in D, E, F. If DO, EO, FO cut the conic again in P, Q, R, prove AP, BQ, CR concurrent.

Solutions (I.) *by* J. H. TAYLOR, M.A.; (II.) *by* H. M. TAYLOR, M.A., F.R.S.

(I.) Project the conic into a circle with the projection of O as its centre. [Small letters are used for the corresponding points on the projection.] Then p, q, r are diametrically opposite to d, e, f. Now, since $cd \cdot ae \cdot bf = ce \cdot af \cdot bd$; ad, be, cf are concurrent. Therefore the equations of the lines ad, be, cf; o being the origin multiplied by readily determined factors and added, vanish identically; and the equations of ap, bq, cr

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differing from those only in change of the signs of the coordinates of d, e, f must, therefore, in like manner, vanish identically; *i.e.*, ap, bq, cr are concurrent. Therefore also AP, BQ, CR are concurrent.

(II.) If the equation of the conic be $ax^2 + by^2 = 1$ and x_1, y_1; x_2, y_2; x_3, y_3 be the coordinates of the points D, E, F respectively, then the equations of the three lines AD, BE, CF respectively are

$$(ax_2x + by_2y - 1)(ax_3x_1 + by_3y_1 - 1) = (ax_3x + by_3y - 1)(ax_1x_2 + by_1y_2 - 1),$$
$$(ax_3x + by_3y - 1)(ax_1x_2 + by_1y_2 - 1) = (ax_1x + by_1y - 1)(ax_2x_3 + by_2y_3 - 1),$$
$$(ax_1x + by_1y - 1)(ax_2x_3 + by_2y_3 - 1) = (ax_2x + by_2y - 1)(ax_3x_1 + by_3y_1 - 1).$$

These equations prove the well known theorem that AD, BE, CF are concurrent.

If O be the *centre* of the conic, then the equations of the lines AP, BQ, CR are

$$(ax_2x + by_2y - 1)(ax_3x_1 + by_3y_1 + 1) = (ax_3x + by_3y - 1)(ax_1x_2 + by_1y_2 + 1),$$
$$(ax_3x + by_3y - 1)(ax_1x_2 + by_1y_2 + 1) = (ax_1x + by_1y - 1)(ax_2x_3 + by_2y_3 + 1),$$
$$(ax_1x + by_1y - 1)(ax_2x_3 + by_2y_3 + 1) = (ax_2x + by_2y - 1)(ax_3x_1 + by_3y_1 + 1).$$

These equations prove that AP, BQ, CR are concurrent when the point O is the centre of the conic; the theory of projection shows that the theorem is true when the position of the point O is unrestricted.

As any point may be projected into the centre of a conic, the problem is true also when the point O is any point in the plane of the conic.

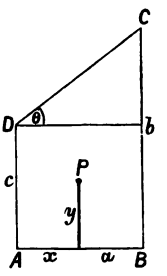
14966. (Rev. W. Allen Whitworth.)—If a point P be taken at random within a trapezium ABCD, the expectation of the area PAB is

$$\tfrac{1}{3}\mathrm{ABC}\,.\,\mathrm{ABD}\,\{1/\mathrm{ABC} + 1/\mathrm{ABD} - 1/\mathrm{ABCD}\}.$$

Solution by R. Chartres *and* John Prescott, B.A.

Taking for simplicity the angles A and B as right angles, mean value of the n-th power of the area

$$\mathrm{APB} = \frac{a^n}{2^n}\int_0^a\int_0^{c+x\tan\theta} y^n dx\,dy \div \int_0^a\int_0^{c+x\tan\theta} dx\,dy$$
$$= \frac{a^n}{2^{n-1}(n+1)(n+2)}\,\frac{b^{n+2}-c^{n+2}}{b^2-c^2}.$$



If $n = 1$, result is

$$\tfrac{1}{6}a\left(\frac{b^2+bc+c^2}{b+c}\right), \text{ or } \tfrac{1}{3}\left(\mathrm{ABC} + \mathrm{ABD} - \frac{\mathrm{ABC}.\mathrm{ABD}}{\mathrm{ABCD}}\right).$$

If $n = 2$, result is

$$\tfrac{1}{24}a^2(b^2+c^2) \quad \text{or} \quad \tfrac{1}{8}(\mathrm{ABC}^2 + \mathrm{ABD}^2).$$

If $c = 0$, the results hold for a triangle.

If $c = b$, the results hold for a rectangle.

The Question as set follows at once, since the mean position of P is the centre of gravity of the figure.

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Solution of $x^3+y^3=17z^3$ *in positive integers.* By Lt.-Col. ALLAN CUNNINGHAM, R.E.

In his *Mémoire sur la Résolution en nombres entiers de l'Équation* $aX^m+bY^m=cZ^n$ (Art. 10), M. DESBOVES gives *general* formulæ (*two* sets) for the solution of the more general equation $X^3+Y^3=cZ^3$ in integers (not necessarily +) when *one* solution (say x, y, z) is *known*. These are

i. $X=x(x^3+2y^3)$, $Y=-y(2x^3+y^3)$, $Z=z(x^3-y^3)$*;

ii. $X=x^9-y^9+3x^3y^3(2x^3+y^3)$, $Y=-x^9+y^9+3x^3y^3(2y^3+x^3)$,
$Z=3xyz(x^6+x^3y^3+y^6)$.

Formulæ i. are said to be due to PRESTET and EULER, and formulæ ii. to ED. LUCAS.

These operations may evidently be repeated and combined, thus leading to an infinity of solutions (usually increasing in magnitude *very rapidly*). If x, z be both + and $x>y$ (numerically), the sign of Y will usually be *opposite* to that of y; and will change at each solution.

Ex.—$x=18$, $y=-1$, $z=7$ is a *known* solution.
The first set of formulæ give

$$X=18(18^3-2)=104940, \quad Y=(2.18^3-1)=11663,$$
$$Z=7(18^3+1)=40831,$$

which is the *lowest* solution in + integers obtainable in this way (this solution had already been obtained by Mr. R. F. DAVIS by a similar process). The results from formulæ ii. run very high.

Further Note on the above equation.

Mr. R. W. D. CHRISTIE writes:—The late Prof. SYLVESTER states (see Quest. 2431) that the simplest representation of 17 as two cubes is

$$\left(\frac{104940}{40831}\right)^3+\left(\frac{11663}{40831}\right)^3=17.$$

Consequently $(104940z)^3+(11663z)^3=17(40831z)^3$
for all values of z.

12904. (Professor MORLEY.)—The tangents at the points where any tangent of a hypocycloid of class 4 (the so-called four-cusped hypocycloid) meets the curve again meet on the cusp circle.

Solution by W. J. GREENSTREET, M.A.

Let $t_0, t_1, \ldots, t_4$ be the values of t at the point of contact and the four points of intersection. The curve has for equation $x^{\frac{1}{3}}+y^{\frac{1}{3}}=a^{\frac{1}{3}}$. The tangent is $(y+t_0^2)\sqrt{(1+t_0^2)}-at_0=0$ .....(1).

* The value of Z is misprinted $Z=z(x^3+y^3)$ in the original.

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Any one of the four points is $[a/(1+t_1^2)^{\frac{1}{2}},\ at_1^3/(1+t_1^2)^{\frac{1}{2}}]$. Substituting and arranging, $(t_1-t_0)^2(t_1^4+2t_1^3t_0+2t_1t_0+t_0^2)=0$,

the first bracket corresponding to the point of contact. But, if (x, y) be a point on the cusp circle $x^2+y^2-a^2=0$, then the equation (1) is

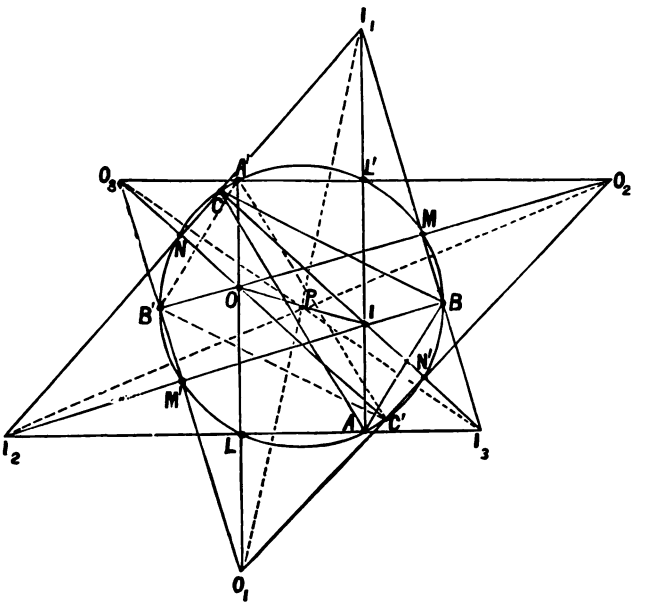
$$t^4+2t^3y/x+2ty/x+y^2/x^2=0;$$

or, writing $y=xt_0$, this becomes $t^4+2t_0t^3+2t_0t+t_0^2=0$, showing that the points of contact of tangents from (x, y) are on one of the two tangents $y=xt_0$. Therefore, &c.

Note to Quest. 14761. *By* R. Biggs.

The following suggests itself as a full statement of the properties involved :—

The circumcentres O, O_1, O_2, O_3 of the four triangles of any orthocentric system I, I_1, I_2, I_3 form a second orthocentric system congruent and parallel to the first, and so placed that I, I_1, I_2, I_3 are the circumcentres of *its* four triangles. The two systems are in perspective at P, the centre of a circle which is the nine-point circle of both. This may be called the twelve-point circle of the twin system, as it passes through L, M, N, L′, M′, N′, the middle points (common in pairs) of the twelve



joins, and through A, B, C, A′, B′, C′, the six pedal points. Given this centre P (which is the circumcentre of the original triangle ABC in the

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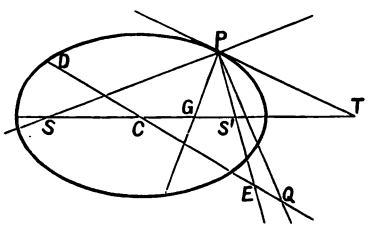
question, and of its analogue A'B'C', *i.e.*, of the pedal triangles of the two systems), and any two points of one system, or one point in each, other than two corresponding ones, the whole figure can be constructed much as in the solutions given, or more readily by the property that the radius of the eight equal circumcircles is double that of the twelve-point circle. The actual question is, in fact, equivalent to "Given base and nine-point centre, construct triangle."

This twelve-point circle touches, of course, the thirty-two in- and ex-circles of the triangles composing the two orthocentric systems; and A, A'; L, L'; &c., are diametrically opposite points.

15015. (Rev. T. Roach, M.A.)—If PQ, perpendicular to SP in the ellipse, meets CD in Q and S'P meets CD in E, show that the normal at P will touch the circle about PEQ in P.

Solution by R. Tucker, M.A.; G. G. Morrice, M.A., M.D.; H. L. Trachtenberg; *and many others.*



$\angle SPQ$ = right angle
= GPT,
and $\angle S'PG = SPG = TPQ$
= PQE;
therefore, &c.

15014. (Professor Nanson.)—A conic touches BC, CA, AB in D, E, F. If DO, EO, FO cut the conics again in P, Q, R, prove AP, BQ, CR concurrent.

Solution by A. M. Nesbitt, M.A.

If the conic be $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$ and the coordinates of O be a, b, c, it is easy to show* that the equation to AP is $(s-b)^2 y = (s-c)^2 z$, where $2s \equiv a+b+c$. Hence the truth of the theorem is obvious.

[* For equation to DO is $x(b-c) = a(y-z)$ and, if we take $y+kz=0$ for equation to AP, the condition that these should meet on the conic is

$$\begin{vmatrix} 1 & -1 & -1 & b-c & 0 \\ -1 & 1 & -1 & -a & 1 \\ -1 & -1 & 1 & a & k \\ b-c & -a & a & 0 & 0 \\ 0 & 1 & k & 0 & 0 \end{vmatrix} = 0,$$

which at once gives $k=-1$, or

$$k = -\left(\frac{a+b-c}{a-b+c}\right)^2.$$]

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14659. (D. Biddle.)—N being any odd number that is prime to 3 and 7, prove that $N^6 \equiv 1 \pmod{168}$.

Solution by Rev. T. Wiggins, B.A.; Aletrop; *and* Lt.-Col. Allan Cunningham, R.E.

By Fermat's theorem (C. Smith, § 380, or Mathews, p. 15), $N^{7-1}-1 = M(7)$, when N is prime to 7, and $N^{3-1}-1 = M(3)$, when N is prime to 3. Also when N is odd $N^2-1 = M(8)$; therefore

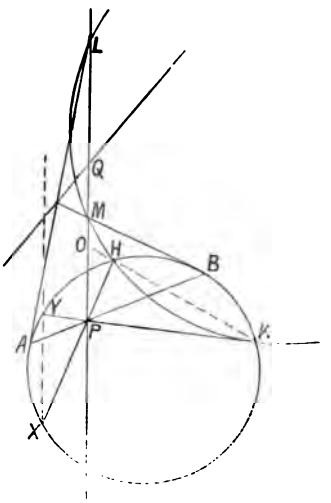
$$N^6-1 = M(7) \quad \text{and} \quad (N^2-1)(N^4+N^2+1) = M(24);$$

therefore $N^6-1 = M(168)$ or $N^6 \equiv 1 \pmod{168}$.

10824. (J. L. Mackenzie, B.A., B.Sc.)—The tangents at points A and B on a circle, and their chord of contact, cut a given line in L, M, P respectively. Any circle through L and M cuts the given circle in H and K, and PH, PK cut the given circle in X and Y; prove XY parallel to LM.

Solutions (I.) *by* J. C. St. Clair; (II.) *by* A. M. Nesbitt, M.A.

(I.) The sides and diagonals of the inscribed quadrilateral HKXY form on the line LMP an involution of which the double points are P and the point Q in which the line meets the polar of P with respect to the circle. In this involution the points in which LM meets the circle are conjugates, as also are L, M, which lie on the tangents at the extremities of the chord AB passing through P. Hence the centre of the involution is the intersection O of the line LM with HK, the common chord of the two circles; and the opposite side XY meets LM in the conjugate of O, namely, at infinity. Therefore XY is parallel to LM.



(II.) Conics through four points cut any transversal in involution. Consider the three following "conics":—the circle AHBK, the line pair (AL, BM), and the coincident line pair (AB, AB). These all pass through AABB. Hence they cut LM in involution, and thus P is a double point on the involution GFLM. But, if KH meet LM in O, O is plainly the centre of the involution, since KH is radical axis, and therefore

$$OG.OF = OL.OM = OH.OK;$$

and thus each rectangle $= OP^2$. Thus OP touches the circle KPH; therefore $\angle OPH = HKP = HXY$; therefore XY is parallel to OP.

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14920. (Aletrop.)—Généralisation du théorème de Fermat. Démontrer, *sans se servir d'aucune interprétation géométrique*, que l'expression

$$\Sigma(x) = x^n - \Sigma x^{n/a} + \Sigma x^{n/ab} - \Sigma x^{n/abc} + \ldots,$$

où a, b, c, sont les facteurs premiers de n, est divisible par n, quels que soient n et x. [M. Kantor, dans sa mémoire:—"Wie viele cyclische Gruppen gibt es in einer quadratischen Transformation der Ebene?" (*Annali di Matematica pura ed applicata*, Tom x., p. 64), trouva le théorème de l'énoncé pour le cas $x = 2$; et M. Picquet le démontra en général par d'autres considérations géométriques (*Comptes Rendus*, 1883, Vol. xcvi., p. 1137).]

Solution by J. P. Hjorsn.

Dans une note complémentaire: "Sur la généralisation du théorème de Fermat due à M. Serret," *Comptes Rendus*, 1883, Vol. xcvi. p. 1424, M. Picquet lui-même écrit: "Effectivement M. Serret a énoncé le même théorème et en a donné la démonstration dans son *Algèbre supérieure*," et cette démonstration est une algébrique. Voir aussi une démonstration de M. Ed. Lucas dans le même volume, p. 1300.

12174. (Rev. T. C. Simmons, M.A.)—Three or more circular wafers of different sizes are fastened at random on the surface of a cylinder. Find the probability that they will all lie (1) on the same generating line, (2) on a generating line drawn at random; also (3) prove that (2) may be reduced to a particular case of (1).

Solution by Rev. Prebendary Whitworth, M.A.

Let s be the circumference of the cylinder, α, β, γ, ... the diameters of the n wafers. If we replace the curved section of the cylinder by a polygon of sm equal sides, so that the diameters of the wafers, in terms of one of the equal sides, are αm, βm, γm, ..., we find the chance of all the wafers covering the same side of the polygonal cylinder to be

$$\{\alpha\beta\gamma \ldots - (\alpha - 1/m)(\beta - 1/m)(\gamma - 1/m)\ldots\} \div s^{n-1}.$$

Hence, ultimately, when m is infinite the required probability (1) is

$$(1/\alpha + 1/\beta + 1/\gamma + \ldots)\,\alpha\beta\gamma \ldots \div s^{n-1}.$$

The probability (2) may be written down at sight, viz., $\alpha\beta\gamma \ldots \div s^n$.

(3) It is obvious that (2) may be obtained from (1) by introducing one more wafer of diameter zero.

15042. (H. Bateman.)—Eliminate A, B, C from the equations

$$\sin 2A \sin A = l \sin B \sin C, \qquad \sin 2B \sin B = m \sin C \sin A,$$
$$\sin 2C \sin C = n \sin A \sin B. \qquad A + B + C = \pi.$$

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Solution by Professor SANJÁNA, M.A.; Rev. T. ROACH, M.A.; *and others*.

We readily get $lmn = 8 \cos A \cos B \cos C$; also

$$(l+m+n)\sin A \sin B \sin C$$
$$= \Sigma \sin 2A \sin^2 A = \tfrac{1}{2}\Sigma \sin 2A (1-\cos 2A) = \tfrac{1}{2}\Sigma \sin 2A - \tfrac{1}{4}\Sigma \sin 4A$$
$$= \tfrac{1}{2}(4 \sin A \sin B \sin C) + \tfrac{1}{4}(4 \sin 2A \sin 2B \sin 2C),$$

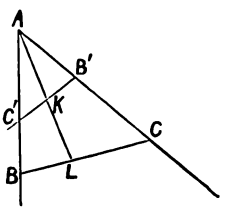
whence $$l+m+n = 2+lmn.$$

14764. (Professor NEUBERG.)—Construire un triangle ABC, étant donnés les côtés indéfinis AB, AC et le centre K des symédianes.

Solutions (I.) *by* Professor JAN DE VRIES, H. W. CURJEL, M.A., *and* R. F. DAVIS, M.A.; (II.) *by* J. BLAIKIE, M.A.; (III.) *by* Professor SANJÁNA, M.A.; (IV.) *by the* PROPOSER.

(I.) AZ being the isogonal conjugate of AK with respect to the given angle XAY, from a point N on AZ draw parallels to AX and AY meeting AY and AX in C′ and B′. If K′ is the symmedian point of AB′C′, parallels to B′K′ and C′K′ passing through K will cut AX, AY in the angular points B, C of the triangle.

(II.) On p. 73 of *The Companion to the Weekly Problem Paper*, Mr. DAVIS has shown that the symmedian point is the centre of gravity of weights a^2, b^2, c^2 placed at the angular points. If, then, AK be produced to L, so that $KL/AK = a^2/(b^2+c^2)$, L is a point in BC. Through K draw C′B′ so as to be bisected at K. The line B′C′ is antiparallel to BC, and the sides a', b', c' of AB′C′ are proportional to a, b, c; therefore



$$a^2/(b^2+c^2) = a'^2/(b'^2+c'^2),$$

and L is known. Through L draw BC antiparallel to B′C′, and the triangle is constructed.

(III.) Let β, γ be the known perpendiculars from K on AC, AB; then $\frac{1}{2}b \tan\omega = \beta$, $\frac{1}{2}c \tan\omega = \gamma$. But

$$\tan\omega = \frac{4\Delta}{a^2+b^2+c^2} = \frac{bc \sin A}{b^2+c^2-bc\cos A};$$

substituting, we have the two equations

$$2\beta(b^2+c^2-bc\cos A) = b^2c \sin A, \quad 2\gamma(b^2+c^2-bc\cos A) = bc^2 \sin A,$$

whence b and c are to be found, β, γ, A being given. We obtain easily

$$c = \frac{2(\beta^2+\gamma^2-\beta\gamma\cos A)}{\beta \sin A}, \quad b = \frac{2(\beta^2+\gamma^2-\beta\gamma\cos A)}{\gamma \sin A}.$$

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Measuring these lengths on AB, AC, and joining the points obtained, the triangle is constructed.

(IV.) Soient P, Q, R les projections de K sur BC, CA, AB; K étant le centre de gravité du triangle PQR, on joint K au milieu S de QR et on prend sur la droite SK une longueur KP = 2SK, &c.

14575. (J. A. THIRD, M.A., D.Sc.)—If A'B'C' be the first BROCARD triangle of ABC, and D, E, F be the points where B'C', C'A', A'B' meet BC, CA, AB respectively, show (1) that DA', EB', FC' are tangents to the BROCARD circle of ABC, and (2) that DA, EB, FC are tangents to the circumconic of ABC which passes through the BROCARD points.

Solution by C. E. YOUNGMAN, M.A.

The BROCARD points Ω, Ω' are concyclic with A'B'C', and AC', BA', CB' meet at Ω, and AB', BC', CA' at Ω'; therefore, by the PASCAL hexagon $A'A'\Omega B'C'\Omega'$, A'A' and B'C' meet on BC; *i.e.*, the tangent at A' goes through D. Similarly, by the hexagon $AA\Omega'BC\Omega$ on the conic $ABC\Omega\Omega'$, AA and BC meet on B'C'.

6021. (E. W. SYMONS, B.A.)—Prove that (1) the cosine of the angle between the planes $Ax + By + Cz = 0$, $A'x + B'y + C'z = 0$, referred to oblique axes inclined at angles α, β, γ, is

$$\frac{AA'.\sin^2\alpha + \ldots - (BC' + B'C)(\cos\alpha - \cos\beta\cos\gamma) - \ldots}{[A^2\sin^2\alpha + \ldots - 2BC(\cos\alpha - \cos\beta\cos\gamma) - \ldots]^{\frac{1}{2}} \times [A'^2\sin^2\alpha + \ldots - 2B'C'(\cos\alpha - \cos\beta\cos\gamma)]^{\frac{1}{2}}};$$

and (2) the length of the perpendiculars from the point (xyz) on the plane $Ax + By + Cz + D = 0$ is

$$(Ax + By + Cz + D)\left(\frac{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma}{A^2\sin^2\alpha + \ldots - 2BC(\cos\alpha - \cos\beta\cos\gamma)}\right)^{\frac{1}{2}}.$$

Solution by Professor NANSON.

Taking two sets of four lines and denoting by (pq) the cosine of the angle between the p-th line of the first set and the q-th line of the second, we find by compounding the arrays formed with the two sets of direction cosines that $|(pq)| = 0$. Taking the first three lines in each set to coincide with the oblique axes, we get

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta & \lambda \\ \cos\gamma & 1 & \cos\alpha & \mu \\ \cos\beta & \cos\alpha & 1 & \nu \\ \lambda' & \mu' & \nu' & \cos\theta \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

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where α, β, γ are the angles between the axes, λ, μ, ν; λ', μ', ν' are the cosines of the angles made with the oblique axes by any two lines, and θ is the angle between the two lines. Solving for $\cos\theta$, we get

$$-\delta\cos\theta = \begin{pmatrix}\lambda & \mu & \nu\\ \lambda' & \mu' & \nu'\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where δ is the minor of $\cos\theta$ in (1), and $\begin{pmatrix}\lambda & \mu & \nu\\ \lambda' & \mu' & \nu'\end{pmatrix}$ is the result obtained by replacing $\cos\theta$ by zero in the first member of (1).

Making the two lines coincide, we get, from (2),

$$-\delta = \begin{pmatrix}\lambda\mu\nu\\ \lambda\mu\nu\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which is the relation connecting the oblique direction cosines λ, μ, ν of a line with α, β, γ.

Taking the two lines in (2) to be perpendicular to the given planes, we have λ, μ, ν proportional to A, B, C, and λ', μ', ν' proportional to A′, B′, C′. Hence from (2), (3), we get

$$-\cos\theta = \begin{pmatrix}A & B & C\\ A' & B' & C'\end{pmatrix}\left\{\begin{pmatrix}ABC\\ ABC\end{pmatrix}\begin{pmatrix}A'B'C'\\ A'B'C'\end{pmatrix}\right\}^{-\frac{1}{2}},$$

and on expansion of the determinants this gives the first formula in the question.

Again, by projection, the distance of the point (xyz) from the plane $\lambda x+\mu y+\nu z-p=0$ is $\lambda x+\mu y+\nu z-p$. Hence, from (3), it follows that the distance of the point xyz from the plane $Ax+By+Cz+D=0$ is

$$(Ax+By+Cz+D)\,\delta^{\frac{1}{2}}\left\{-\begin{pmatrix}ABC\\ ABC\end{pmatrix}\right\}^{-\frac{1}{2}},$$

and on expansion this gives the second formula in the question.

7791. (J. O'Regan.)—A person names a group of three integers (not necessarily different, but each one not greater than n): find the chance that the integers named will be such that every two are together greater than the third.

Solution by John Prescott, B.A.

The first integer may be chosen in n ways, the second in n ways, and the third in n ways; therefore total number of ways of choosing the three integers (if we consider the same integers, arranged in a different order, as a different choice) is n^3.

Now, if the first number is m, in order that the sum of the other two may be *not* greater than m, their sum may have all the values

$$2,\ 3,\ 4,\ 5,\ 6,\ \ldots,\ m.$$

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When their sum is r, their values may be

1st number.	2nd number.
1	$r-1$
2	$r-2$
3	$r-3$
⋮	⋮
$r-1$	1

i.e., there are $(r-1)$ ways in which their sum may be r. Therefore the total number of ways in which their sum may be not greater than m is

$$1+2+3+4+\ldots+(m-1) = \tfrac{1}{2}[m(m-1)].$$

Therefore, since m may have all values from 1 to n, the total number of ways in which the first number may be greater than the sum of the other two is

$$\sum_{m=1}^{m=n} \frac{m(m-1)}{2} = \frac{n(n+1)(n-1)}{6}.$$

Therefore the number of ways in which any one of the numbers may be greater than the sum of the other two is $\frac{1}{6}[n(n+1)(n-1)]$.

Hence the chance that any one of the numbers will be greater than the sum of the other two is

$$[n(n+1)(n-1)]/2n^3 = (n^2-1)/2n^2,$$

and consequently the chance that any two will be greater than the other is

$$1-(n^2-1)/2n^2 = (n^2+1)/2n^2.$$

Note.—If n be the number of unit lengths in a given straight line, the above result gives the chance that three lines whose measures are p, q, and r respectively (where these quantities are integers, each less than n) may be the sides of a possible triangle. Hence, if the unit be taken infinitely small, the chance that three random lines, each less than P, may form a triangle is given by $\lim_{n=\infty} (n^2+1)/2n^2 = \frac{1}{2}$.

[Mr. CHARTRES gives the same results and refers readers for complete solutions of this and similar problems to Prebendary WHITWORTH's *DCC. Exercises.*]

15040. (R. CHARTRES.)—Find the mean value of the n-th power of the area of a cyclic quadrilateral whose perimeter $(2s)$ is constant, and deduce the results for (1) a kite, (2) a triangle.

Solution by the PROPOSER.

$(\text{Area})^n = \{(s-a)(s-b)(s-c)(s-d)\}^{\frac{1}{2}n}$, where $a+b+c+d = 2s$; therefore

$$\text{mean value} = (2s)^{2n} \iiint x^{\frac{1}{2}n} y^{\frac{1}{2}n} z^{\frac{1}{2}n} w^{\frac{1}{2}n}\, dx\, dy\, dz\, dw \div \iiint dx\, dy\, dz\, dw$$

$$= 3!\,(2s)^{2n} \{\Gamma(\tfrac{1}{2}n+1)\}^4 \div \Gamma\{4(\tfrac{1}{2}n+1)\}.$$

If $d = 0$, we get the case for a triangle

$$= 2!\, s^{2n} \{\Gamma(\tfrac{1}{2}n+1)\}^3 \div \Gamma\{3(\tfrac{1}{2}n+1)\}.$$

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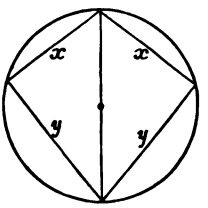
If $a = b$ and $c = d$, we get the case for a kite

$$= s^{2n}\left\{G(n+1)\right\}^2 + G\left\{2(n+1)\right\}.$$

But this can also be obtained in a very elementary way.

We require the mean value of $(xy)^n$ when

$$x + y = s.$$



Now $s^{2n} = (x+y)^{2n}$,

containing $(2n+1)$ terms, each of the same mean value, a theorem first given by Prebendary Whitworth; therefore mean value

$$= \frac{s^{2n}}{2n+1} + \frac{(2n)!}{(n!)^2}.$$

14878. (J. J. Barniville, B.A.)—Having $u_n + u_{n+3} = 3u_{n+1} + 3u_{n+2}$,

prove that

$$\frac{1}{12}+\frac{1}{7}+\frac{1}{60}+\frac{1}{189}+\frac{1}{740}+\ldots = \frac{1}{4},$$

$$\frac{1}{24}+\frac{1}{9}+\frac{1}{104}+\frac{1}{315}+\frac{1}{1248}+\ldots = \frac{1}{6},$$

$$\frac{1}{40}+\frac{1}{11}+\frac{1}{160}+\frac{1}{473}+\frac{1}{1888}+\ldots = \frac{1}{8},$$

$$\frac{1}{15}+\frac{1}{12}+\frac{1}{85}+\frac{1}{276}+\frac{1}{1071}+\ldots = \frac{1}{6},$$

$$\frac{1}{21}+\frac{1}{36}+\frac{1}{175}+\frac{1}{612}+\frac{1}{2325}+\ldots = \frac{1}{12},$$

$$\frac{1}{20}+\frac{1}{39}+\frac{1}{180}+\frac{1}{637}+\frac{1}{2412}+\ldots = \frac{1}{12}.$$

Solutions (I.) *by* H. W. Curjel, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) If $\omega_n = \dfrac{\alpha x^n + (-1)^n \beta x^{-n}}{\alpha x^n - (-1)^n \beta x^{-n}}$, where $x > 1$, so that $\omega_\infty = 1$, and $(n+1)$-th term of series

$$= \frac{1}{a^2x^{2n} + \beta^2 x^{-2n} + (-1)^n \alpha\beta\,(x^2 - x^{-2})} = \frac{\omega_{n+1} - \omega_{n-1}}{2(-1)^n \alpha\beta\,(x^2 - x^{-2})},$$

then sum to n terms $= \dfrac{\omega_0 - \omega_{-1} + (-1)^n(\omega_{n-1} - \omega_n)}{2\alpha\beta\,(x^2 - x^{-2})}$,

and sum to infinity $= \dfrac{\omega_0 - \omega_{-1}}{2\alpha\beta\,(x^2 - x^{-2})}$.

Putting $x = \dfrac{\sqrt{3}+1}{\sqrt{2}}$ in each case, and

$$\alpha = \frac{5+\sqrt{3}}{2\sqrt{3}},\ \frac{7+\sqrt{3}}{2\sqrt{3}},\ \frac{3\sqrt{3}+1}{2},\ \frac{4+\sqrt{3}}{\sqrt{6}},\ \frac{5+2\sqrt{3}}{\sqrt{6}},\ \frac{7+3\sqrt{3}}{2\sqrt{3}},$$

$$\beta = \frac{5-\sqrt{3}}{2\sqrt{3}},\ \frac{7-\sqrt{3}}{2\sqrt{3}},\ \frac{3\sqrt{3}-1}{2},\ \frac{4-\sqrt{3}}{\sqrt{6}},\ \frac{5-2\sqrt{3}}{\sqrt{6}},\ \frac{7-3\sqrt{3}}{2\sqrt{3}},$$

the sum to infinity $= \dfrac{1}{4},\ \dfrac{1}{6},\ \dfrac{1}{8},\ \dfrac{1}{6},\ \dfrac{1}{12},\ \dfrac{1}{12}$.

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(II.) Series (1)

$$= \frac{1}{12}+\frac{1}{60}+\frac{1}{740}+\frac{1}{10212}+\dots+\frac{1}{7}+\frac{1}{189}+\frac{1}{2727}+\frac{1}{38077}+\dots$$

$$= \frac{1}{2}\left\{\frac{1}{2.3}+\frac{1}{3.10}+\frac{1}{10.37}+\frac{1}{37.138}\dots\right\}$$

$$+\left\{\frac{1}{1.7}+\frac{1}{7.27}+\frac{1}{27.101}+\frac{1}{101.377}\dots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{6-}\frac{2^2.3^2}{3^2.4-}\frac{3^2.10^2}{10^2.4-}\frac{10^2.37^2}{37^2.4-}\dots\right\}+\left\{\frac{1}{7-}\frac{7^2}{7^2.4-}\frac{7^2.27^2}{27^2.4-}\dots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{6-}\frac{4}{4-}\frac{1}{4-}\frac{1}{4-}\dots\right\}+\left\{\frac{1}{7-}\frac{1}{4-}\frac{1}{4-}\frac{1}{4-}\dots\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{6-4(2-\sqrt{3})}\right\}+\frac{1}{7-(2-\sqrt{3})} = \frac{1}{4(2\sqrt{3}-1)}+\frac{1}{5+\sqrt{3}}$$

$$= \frac{2\sqrt{3}+1}{44}+\frac{10-2\sqrt{3}}{44} = \frac{1}{4}.$$

Series (2)

$$= \frac{1}{24}+\frac{1}{104}+\frac{1}{1248}+\frac{1}{17184}+\dots+\frac{1}{9}+\frac{1}{315}+\frac{1}{4585}+\dots$$

$$= \frac{1}{2}\left\{\frac{1}{3.4}+\frac{1}{4.13}+\frac{1}{13.48}+\frac{1}{48.179}+\dots\right\}+\frac{1}{1.9}+\frac{1}{9.35}+\frac{1}{35.131}+\dots$$

$$= \frac{1}{2}\left\{\frac{1}{12-}\frac{3^2.4^2}{4^2.4-}\frac{4^2.13^2}{13^2.4-}\frac{13^2.48^2}{48^2.4-}\dots\right\}+\frac{1}{9-}\frac{9^2}{9^2.4-}\frac{9^2.35^2}{35^2.4-}\dots$$

$$= 2\left\{\frac{1}{12-9(2-\sqrt{3})}\right\}+\frac{1}{9-(2-\sqrt{3})} = \frac{1}{6(3\sqrt{3}-2)}+\frac{1}{7+\sqrt{3}}$$

$$= \frac{3\sqrt{3}+2}{138}+\frac{21-3\sqrt{3}}{138} = \frac{1}{6}.$$

The other four series may be similarly treated, but I have not tested the results.

[N.B.—The last result in Mr. Barniville's Quest. 14344 (Vol. lxxiv.) is quite correct, as I again find. Mr. Woodall's solution is correct but for a slip in the very last line, where $\frac{1}{2+2(\sqrt{3}-1)}$ should occur instead of $\frac{1}{2+}\ \frac{2}{\sqrt{3}-1}$.]

14642. (Robert W. D. Christie.)—Establish directly, by use of the quintic roots of minus unity, Dr. Reuschle's *Einfach Factor* of the prime 11, and, if possible, extend same to other primes.

Solution by the Proposer.

$$r_1 = -r^{6l-3}+r^{6l-4}-r^{6l-5}\dots+1 \bmod 12l-1 \quad (l=1),$$

$$r_1 = -r^3+r^2-r^1+1 \bmod 11.$$

Let r_1, an associated root of r, $= 2$; then, $r^3-r^2+r+1 = 0 \bmod 11$. Take $r = \omega_n$ any quintic root of unity; therefore $\omega_4-\omega_3+\omega+1 = 0$, *i.e.*, $2\omega_3+\omega_2 = \omega^2+2\omega^4$, which is Reuschle's *Einfach*.

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14962. (Rev. T. Roach, M.A.)—Tom, Dick, and Harry set out on a journey into the wilderness carrying a small flag which is to be carried forward as many days' journey as possible. Tom and Dick carry 60 rations each and Harry carries 30. Tom and Dick eat 3 rations a day each and Harry eats $1\frac{1}{2}$. They can put down any part of their loads, to be taken up on their return, but cannot return for a second load. Find the farthest point to which they can get the flag.

Solution by J. Blaikie, M.A., *and the* Proposer.

During each day Tom and Dick eat 3 rations and Harry $1\frac{1}{2}$. At the end of each day Tom and Dick put down 3 rations and Harry $1\frac{1}{2}$ to serve for their return. Thus in each day 15 rations are disposed of. After 4 days Tom has 36 rations left; he gives 24 to Dick and 12 to Harry, and sets off on his return journey, picking up the rations he dropped before. Dick has now 60 rations and Harry has 30. In each day 9 rations are disposed of. After $6\frac{2}{3}$ days Dick has 20 rations left, which he gives to Harry, who has now 30 rations, and is therefore able to go 10 days' further journey. The furthest point reached is therefore $20\frac{2}{3}$ days' journey.

14725. (Robert W. D. Christie.)—Prove the following theorems on circulators. P = any prime having an *even* period; H_1 = the number formed by the figures of the first half; H_2 = the number formed by the figures of the second half. Then

$$P-1 = (H_2+1)/(H_1+1), \quad H_2(P-1) = (P-2)\,10^{\frac{1}{2}P} + H_1 \ldots(1, 2),$$

$$(H_2+1)/(H_1+1) = (10^{\frac{1}{2}P}-H_1)/(10^{\frac{1}{2}P}-H_2), \quad 10^{\frac{1}{2}P}-H_1 = 0 \bmod P-1 \ (3, 4),$$

$$H_1+H_2+2 = 0 \bmod P = 10^{\frac{1}{2}P}+1 \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

$$(H_1+H_2+2)/(H_2-H_1) = P/(P-2) \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

$$P\,(H_1+1) = 10^{\frac{1}{2}P}+1 \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and various others by combination.

Example of (1)— $P = 7, \ \frac{1}{7} = 142857;$

therefore $7-1 = (857+1)/(142+1) = 6.$

Solution, with addition to same, by the Proposer.

Of the prime P let the period be $2p$. Then from the identity we get

$$10^{2p}-1/P \equiv 10^p\left\{(10^p+1)/P-1\right\} + \left\{(P-1)\,10^p-1\right\}/P \equiv 10^p h_1 + h_2;$$

thus $h_1+1 = (10^p+1)/P$ and $h_2+1 = \left\{(P-1)(10^p+1)\right\}/P;$

therefore $(h_2+1)/(h_1+1) = P-1 \ldots\ldots\ldots\ldots\ldots\ldots(1).$

Again, since $h_2 = 10^p-(10^p+1)/P, \ \therefore\ 10^p-h_2 = (10^p+1)/P = h_1+1;$

and since $h_1 = (10^p+1)/P-1, \ \therefore\ 10^p-h_1 = \left\{(P-1)(10^p+1)\right\}/P = h_2+1;$

therefore $(10^p-h_1)(h_1+1) = (10^p-h_2)(h_2+1) \ldots\ldots\ldots\ldots\ldots\ldots(2).$

Also, from the identity and (1),

$$h_2(h_2+1)/(h_1+1) \equiv \left\{(h_2+1)/(h_1+1)-1\right\}(h_1+h_2+1)+h_1,$$

we get $h_2(P-1) \equiv (P-2)\,10^p + h_1 \ldots\ldots\ldots\ldots\ldots\ldots(3),$

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and, since (1), $(h_2+1)/(h_1+1) = p-1$, therefore

$$(h_1+h_2+2)/(h_2-h_1) = p/(p-2) \quad \ldots\ldots(4).$$

Other results necessarily follow. An interesting one is

$$(P-1)\,10^p - P(h_2) = 1.$$

Thus 10^p cannot contain any prime in h_2, nor h_2 any prime in $P-1$ ...(5). Similar conclusions may be drawn from (2) *supra*.

14652. (Professor E. Lemoine.)—(1) Le point Φ du triangle ABC, dont les coordonnées barycentriques sont $a^2b^2+a^2c^2-b^2c^2$, $b^2c^2+b^2a^2-c^2a^2$, $c^2a^2+c^2b^2-a^2b^2$, et qui jouit, d'ailleurs, de remarquables propriétés, est à l'intersection de la droite qui joint le milieu de la distance des points de Brocard au barycentre, et sur chacune des trois droites qui joignent le symétrique d'un sommet par rapport au milieu du côté opposé, au pied, sur le côté, de la symédiane partant du sommet opposé. (2) Soient A_1, B_1, C_1 les points où $A\Phi$, $B\Phi$, $C\Phi$ coupent BC, CA, AB ; les parallèles à AB et à AC menées par A_1 coupent respectivement AC et AB en μ et ν' ; les parallèles à BC et à BA menées par B_1 coupent respectivement BA et BC en ν et λ' ; les parallèles à CA et à CB menées par C_1 coupent respectivement BC et BA en λ et μ'. Cela posé, les points λ, μ, ν, λ', μ', ν' sont concycliques.

Solution by C. E. Youngman, M.A.

Supposing Φ any point in ABC, let us find its locus when μ, μ', ν, ν' are concyclic, *i.e.*, when

$$A\mu \,.\, A\mu' = A\nu \,.\, A\nu'.$$

We have $A\mu : AC = BA_1 : BC$,

or $\qquad a \,.\, A\mu = b \,.\, BA_1$;

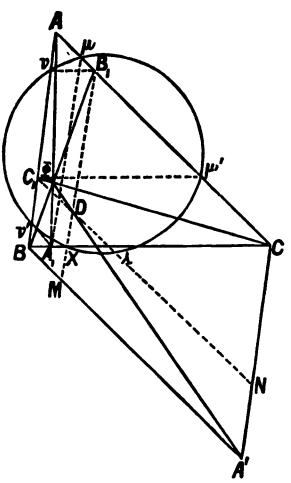
and similarly

$$c \,.\, A\mu' = b \,.\, AC_1, \quad b \,.\, A\nu = c \,.\, B_1A,$$

and $\qquad a \,.\, A\nu' = c \,.\, A_1C.$

Hence $b^3 \,.\, BA_1 \,.\, AC_1 = c^3 \,.\, B_1A \,.\, A_1C.$

But $BA_1 \,.\, CB_1 \,.\, AC_1 = B_1A \,.\, C_1B \,.\, A_1C$; therefore $BC_1 : CB_1 = c^3 : b^3$.



Complete the parallelogram BACA′ (so that A′ is the image of A for the middle of BC), and let $B_1\lambda'$, BA′ meet at M, $C_1\lambda$ and CA′ at N, and $B_1\lambda'$, $C_1\lambda$ at D. Then

DM : DN = BC_1 : CB_1 = const. ;

therefore D moves on a fixed line A′D, which is also the locus of Φ, because BB_1, A′D, and CC_1 concur, being the diagonals of three adjacent parallelograms AM, MN, NA. And

$$B\lambda : BC = BC_1 : BA, \text{ and } BC : C\lambda' = AC : CB_1 ;$$

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therefore $B\lambda : C\lambda' = b \cdot BC_1 : c \cdot CB_1 = c^2 : b^2$;

therefore, when D is on BC (and λ, λ' coincide with it), AD is a symmedian of ABC.

We see also that A'D and the corresponding loci B'E and C'F will divide the sides of A'B'C' in ratios $b^2 : c^2$, $c^2 : a^2$, $a^2 : b^2$, whose product is unity; therefore they meet at a point Φ for which $\mu\mu'\nu\nu'$, $\nu\nu'\lambda\lambda'$, and $\lambda\lambda'\mu\mu'$ will be circles—and not three circles, but one and the same, for, if distinct, the radical axes $\lambda\lambda'$, $\mu\mu'$, $\nu\nu'$ would not concur.

And Φ is the mean centre of A'B'C' for multiples $a^{-2} : b^{-2} : c^{-2}$. Now the Brocard points of A'B'C' are the mean centres for multiples $b^{-2} : c^{-2} : a^{-2}$ and $c^{-2} : a^{-2} : b^{-2}$; therefore these three points have the same centroid G as A'B'C' or ABC, and ΦG bisects the join of the Brocard points in A'B'C', and consequently in ABC also.

14839. (Professor Neuberg.) — Dans une parabole on mène la symétrique de la normale par rapport à l'ordonnée. Trouver l'enveloppe de cette droite.

Solution by Professor Sanjána, M.A.; *and* Constance I. Marks, B.A.

Let the normal make an acute angle θ with the axis, and let $\tan\theta = t$; then any point on the parabola is $(at^2, 2at)$; and the equation of the symmetric becomes $y - 2at = t(x - at^2)$. This being written

$$at^3 - t(2a + x) + y = 0,$$

we see that, for t variable, the envelope is $27ay^2 = 4(2a + x)^3$, a semicubical parabola.

15061. (J. Prescott.)—From the fact that the inverse of a plane is a sphere through the centre of inversion, deduce that the inverse of a circle, with respect to a point not in its plane, is a circle.

Solution by the Proposer.

Let A be a plane whose inverse with respect to O is a sphere B passing through O. Let OM be normal to A. It is evident from symmetry that the centre of B lies in OM.

Now, B being the inverse of A, it follows, since the relation is a reciprocal one, that A is the inverse of B. Consequently, by a proper choice of the constant of inversion, any sphere passing through O and having its centre in OM can be made to invert into the plane A. Therefore, with *any* constant of inversion, such a sphere will invert into a plane parallel to A, *i.e.*, parallel to the tangent plane at O. Hence, the inverse of any sphere whatever, with respect to a point on its surface, is a plane parallel to the tangent plane at that point.

Now let C be any circle in a plane P, and O any point not in P. Then

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a sphere S can be described having the circle C and the point O on its surface. With respect to O, the inverse of P is a sphere, and the inverse of S is a plane; therefore the inverse of C, the intersection of P and S, is the intersection of their inverses, *i.e.*, a circle.

14974. (R. Chartres.)—In a triangle ABC, if $\cos A = \frac{1}{3}$, and a straight line be drawn from A making an angle $\frac{1}{3}\pi$ with its bisector, then the parts intercepted without and within the in-circle are equal to its radius, and the parts intercepted without and within the ex-circle are equal to its radius.

Solutions (I.) *by* Lionel E. Reay, B.A.; (II.) *by* A. W. Poole, B.A.; (III.) *by the* Proposer.

(I.) $\cos A = \frac{1}{3}$; $\therefore \sin^2 \frac{1}{2}A = \frac{1}{3}$.

$OA = r \operatorname{cosec} \frac{1}{2}A = r\sqrt{3}$;

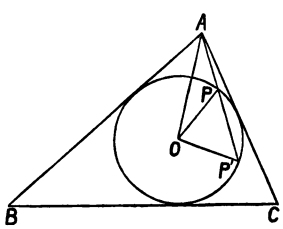
therefore

$\tan OAP' = \tan \frac{1}{3}\pi = 1/\sqrt{3} = OP'/OA$;

therefore AOP′ is a right angle; therefore $\angle OP'A = \frac{1}{3}\pi$;

therefore OPP′ is an equilateral triangle, and PP′ = PO = PA.

For the escribed circle, $I_1A = r_1\sqrt{3}$; therefore, &c.



(II.) $\cos A = \frac{1}{3}$

$= 1 - 2\sin^2 \frac{1}{2}A$;

therefore $\sin^2 \frac{1}{2}A = \frac{1}{3}$;

therefore

$\operatorname{cosec} \frac{1}{2}A = \sqrt{3}$;

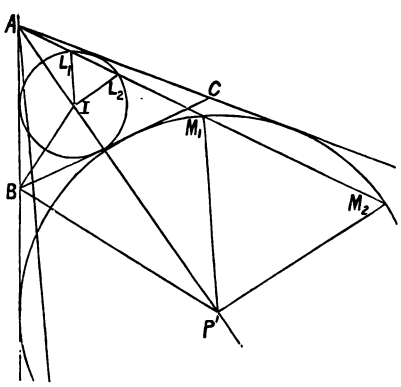
therefore

$AI = r \operatorname{cosec} \frac{1}{2}A = r\sqrt{3}$,

and $AI_1 = r_1\sqrt{3}$.

Consider now the triangle AIL_1, and let $AL_1 = x$; therefore

$IL_1^2 = AI^2 + x^2 - 2x AI \cos \frac{1}{3}\pi$;



therefore

$r^2 = 3r^2 + x^2 - 2x r\sqrt{3}\,\frac{1}{2}\sqrt{3}$; therefore $x^2 - 3rx + 2r^2 = 0$;

therefore $x = r$ or $2r$; *i.e.*, $AL_1 = r$, $AL_2 = 2r$; therefore $L_1L_2 = r$.

Similarly, for triangle AI_1M_1, we have ($AM_1 = y$)

$I_1M_1^2 = AI_1^2 + y^2 - 2y AI_1 \cos \frac{1}{3}\pi$ or $y^2 - 3r_1y + 2r_1^2 = 0$;

therefore $y = r_1$ or $2r_1$, *i.e.*, $AM_1 = M_1M_2 = r_1$.

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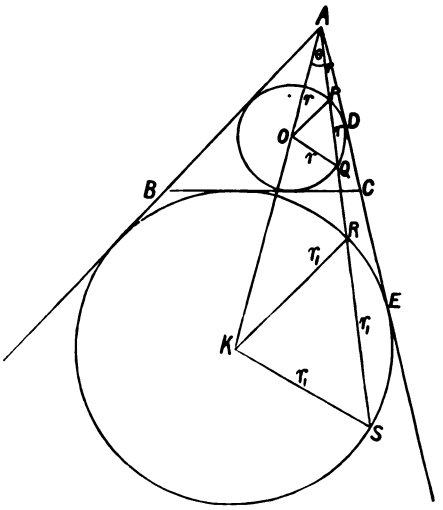
(III.) If AP = PQ = r, evidently $\theta = 30°$. Also rectangle AP . AQ = AD²,



or $2r^2 = r^2 \cot^2 \frac{1}{2}A$, or $\cos A = \frac{1}{3}$. Similarly for the escribed circle.

12602. (M. Brierley.) — There are four numbers w, x, y, z $wxy = 252$, $wxz = 756$, $xyz = 432$: find the numbers.

Solution by W. J. Greenstreet, M.A.

$252 = 7 \times 36$, $756 = 7 \times 108$. If $w = 7$, $xy = 36$, $xz = 108$; therefore

$$z = \tfrac{432}{36} = 12, \quad y = \tfrac{432}{108} = 4, \quad x = 9;$$

therefore 7, 9, 4, 12 are the values required.

14748. (James Blaikie, M.A.)—If a tetrahedron have its opposite edges equal, its four escribed spheres are equal, and the radius of an escribed sphere is equal to twice the radius of the inscribed sphere.

Solution by R. Tucker, M.A., *the* Proposer, *and others.*

The triangular faces are congruent; hence, if A is the area of a face, inscribed radius = $\lambda/4A$, and any escribed radius (where λ is a constant) = $\lambda/(3A - A) = \lambda/2A$; hence, &c.

N.B.—λ = three times the volume of the tetrahedron.

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Elementary Integrals obtained by Calculation and not by Inference.

By GEORGE SCOTT, M.A.

Since, in general, $d(uv)/dx = u\,dv/dx + v\,du/dx$,

the general formula for integration by parts is

$$\int u\,dv/dx\,dx = uv - \int v\,du/dx\,dx.$$

In the integral $\int x^m\,dx$, let $x^m = u$ and $1 = dv/dx$. Then

$$\int x^m\,dx = x^m x - \int x\,mx^{m-1}dx \quad \text{or} \quad \int x^m\,dx = x^{m+1} - m\int x^m\,dx;$$

therefore $(m+1)\int x^m\,dx = x^{m+1}$ or $\int x^m\,dx = x^{m+1}/(m+1)$.

In the case when $m = -1$ this formula fails, but we may proceed as follows:— $\int dx/x = \int dx.\,1/[1-(1-x)]$

$$= \int dx\,[1 + (1-x) + (1-x)^2 + (1-x)^3 + (1-x)^4 + \ldots];$$

therefore $\int dx/x = x - \frac{1}{2}(1-x)^2 - \frac{1}{3}(1-x)^3 - \frac{1}{4}(1-x)^4 - \ldots$

$$= x - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \tfrac{1}{5}(x-1)^5 - \ldots;$$

therefore $\int dx/x = 1 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots$.

Now $x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$ is $\log_e(1+x-1)$, that is, $\log_e x$, and, as the expression above differs from this only in the constant 1, we have, apart from an arbitrary constant, the indefinite integral $\int dx/x$ is equal to $\log_e x$.

If we apply integration by parts to $\int \sin x\,dx$, we have

$$x\sin x - \int x\cos x\,dx = x\sin x - \tfrac{1}{2}x^2\cos x - \int \tfrac{1}{2}x^2\sin x\,dx$$

$$= x\sin x - \tfrac{1}{2}x^2\cos x - \frac{x^3}{2.3}\sin x + \int \frac{x^3}{2.3}\cos x\,dx.$$

Proceeding thus, we get

$$\int \sin x\,dx = \sin x\,[x - x^3/3! + x^5/5! - \ldots] + \cos x\,[-x^2/2! + x^4/4! - \ldots],$$

or

$$\sin x \sin x + \cos x\,(\cos x - 1);$$

therefore $\int \sin x\,dx = \sin^2 x + \cos^2 x - \cos x$ or $1 - \cos x$;

therefore, neglecting an arbitrary constant, the indefinite integral $\int \sin x\,dx$ is equal to $-\cos x$.

$$\int \cos x\,dx = x\cos x + \int x\sin x\,dx = x\cos x + \tfrac{1}{2}x^2\sin x - \int \tfrac{1}{2}x^2\cos x\,dx$$

$$= x\cos x + \tfrac{1}{2}x^2\sin x - \frac{x^3}{2.3}\cos x - \int \frac{x^3}{2.3}\sin x\,dx.$$

Proceeding thus, we have

$$\cos x\,[x - x^3/3! + x^5/5! - \ldots] + \sin x\,[x^2/2! - x^4/4! + x^6/6! - \ldots],$$

or

$$\cos x \sin x + \sin x\,(-\cos x + 1);$$

therefore

$$\int \cos x\,dx = \sin x.$$

The elementary integrals $\int dx/(1+x^2)$, $\int dx/\sqrt{(1-x^2)}$ are obtained at once by the assumption in the first case $x = \tan\theta$, in the second $x = \sin\theta$ or $\cos\theta$.

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9300. (George Heppel, M.A.)—If a whole number, when divided by prime numbers m and n, gives remainders a and b respectively, what will be the remainder when it is divided by mn?

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.; (II.) *by* D. Biddle; (III.) *by* Robert W. D. Christie.

(I.) Let $m > n$; then N is of form $N = \lambda mn + \mu m + a$. Now let $m \equiv m_0 \pmod{n}$, and $a \equiv a_0 \pmod{n}$. Then

$$N \equiv \mu m + a \equiv \mu m_0 + a_0 \equiv b \pmod{n} \quad \text{or} \quad m_0 \mu \equiv b - a_0 \pmod{n}.$$

Since m_0, b, a_0 are *known*, this congruence suffices to give μ. Hence

$$N \equiv \mu m + a \pmod{mn}.$$

(II.) Let $m > n$, and $N \equiv x \pmod{mn}$; then $x < mn$, $x \equiv a \pmod{m}$; $\equiv b \pmod{n}$. Let $x = my + a = ns + b$; then $y = \{ns - (a-b)\}/m$ and $z = (my + a - b)/n$; whence

$$(m-n)\,y \equiv b - a \pmod{n} \quad \text{and} \quad (m-n)\,z \equiv b - a \pmod{m}.$$

Here y and z are the only unknown quantities; moreover, $y < n$, $z < m$. Consequently y, z can be found by trial with comparative ease. Where $m = \mu n + \kappa$, $(m-n)\,y \equiv b - a \pmod{n}$ reduces to $\kappa y \equiv b - a \pmod{n}$.

(III.)
$$x = my + a = nx + b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$= mnv + (cm + a) = mnw + (dn + b),$$

by ordinary division by factors; therefore $cm + a = dn + b$, *i.e.*,

$$dn - cm = a - b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where m, n, a, b are given. Now, solving $dn - cm = 1$ by G.C.M. of m, n, and then multiplying throughout by $(a-b)$ given, we get the values of c, d (eliminating m and n when necessary), and the required remainder is either $cm + a$ or $dn + b$.

Note.—The following particular case shows the method which the Proposer would have used:—Let some number divided by 43 and then by 29 leave partial remainders 14 and x; and when by 29 and 43, 20 and y. Then the complete remainder when divided by 43 and 29 is

$$43x + 14 = 29y + 20, \quad \text{whence} \quad y = x + \tfrac{1}{29}(14x - 6);$$

therefore $\frac{1}{29}(28x - 12)$ or $x - \frac{1}{29}(x + 12)$ is integral. $x = 29t - 12$. Least value is 17. Complete remainder is $43 \times 17 + 14 = 745$.

15078. (Professor Hudson, M.A.)—If a uniform chain of given length be fastened to two points at a given distance apart in the same horizontal line, and be acted on by vertical forces such that the force acting on a length ds is $mgds + \mu g dx$, prove that the horizontal tension is approximately $(m + \mu)\, g \left\{x^3/6\,(s-x)\right\}^{\frac{1}{2}}$, and that the sag at the middle point is $\left\{\frac{1}{8}x(s-x)\right\}^{\frac{1}{2}}$, assuming that $(s-x)/x$ is small.

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Solutions (I.) *by the* PROPOSER; (II.) *by* J. PRESCOTT, B.A.

(I.) Let C be the constant horizontal tension, ϕ the angle which the tangent at (x, y) makes with the horizon, s the length of the arc from the middle point of the chain to (x, y). Then

$$d/ds\,(C\,dy/dx) = mg + \mu g\,dx/ds,$$

whence
$$\left.\begin{aligned} ds/d\phi &= C/g\cos^2\phi\,(m+\mu\cos\phi), \\ dx/d\phi &= C/g\cos\phi\,(m+\mu\cos\phi) && = C/g\,(m+\mu) \\ d\,(s-x)/d\phi &= C\,(1-\cos\phi)/g\cos\phi\,(m+\mu\cos\phi) && = C\phi^2/2g\,(m+\mu) \\ dy/d\phi &= C\sin\phi/g\cos^2\phi\,(m+\mu\cos\phi) && = C\phi/g\,(m+\mu) \end{aligned}\right\},$$

approximately.

Integrating, since x, y, s, ϕ vanish together,

$$x = C\phi/g\,(m+\mu),\quad (s-x) = C\phi^2/6g\,(m+\mu),\quad y = C\phi^2/2g\,(m+\mu);$$

from which $C = (m+\mu)\,g\left\{x^2/6\,(s-x)\right\}^{\frac{1}{2}},\quad y = \left\{\tfrac{3}{2}x\,(s-x)\right\}^{\frac{1}{2}}.$

(II.) Let T be the constant horizontal tension, ψ the angle which the tangent to the chain at any point makes with the horizontal. Let $2s_0$ be the length of the chain, $2x_0$ the distance between the ends, l the sag at the middle. Then

$$T\tan\psi = \int mg\,ds + \int \mu g\,dx = mgs + \mu g x \quad \text{.....(i.),}$$

if s and x are measured from the lowest point. Hence, neglecting the difference between s and x,

$$T^2\tan^2\psi = x^2\,(mg+\mu g)^2,\quad i.e.,\ T^2\left\{(ds/dx)^2-1\right\} = x^2\,(mg+\mu g)^2 \text{...(ii.).}$$

Put $s = x+\theta$, where θ is small compared with x. Then

$$ds/dx = 1 + d\theta/dx.$$

Hence (ii.) becomes

$$T^2\left\{(1+d\theta/dx)^2-1\right\} = x^2\,(mg+\mu g)^2 \text{ or } T^2\left\{2\,d\theta/dx\right\} = x^2\,(mg+\mu g)^2 \text{...(iii.),}$$

neglecting $(d\theta/dx)^2$, since $d\theta/dx$ is a small quantity. Integrating (iii.),

$$2T^2\theta_0 = \tfrac{1}{3}x_0^3\,(mg+\mu g)^2 \text{ or } T = (mg+\mu g)\left\{x_0^3/6\,(s_0-x_0)\right\}^{\frac{1}{2}} \text{...(iv.).}$$

Again, neglecting θ, (i.) may be written

$$T\tan\psi = mgs + \mu gs.$$

Differentiating with respect to ψ,

$$T\sec^2\psi = (mg+\mu g)(ds/d\psi) = (mg+\mu g)\,\rho.$$

But $2\rho = \lim x^2/y$. Hence, at the middle point $2\rho = x_0^2/l$ approximately, and $\sec^2\psi = 1$; therefore $T = (mg+\mu g)(x_0^2/2l)$.

Substituting for T in (iv.), we get

$$\left\{x_0/6\,(s_0-x_0)\right\}^{\frac{1}{2}} = x_0^2/2l;\quad \text{therefore } l = \left\{\tfrac{2}{3}x_0\,(s_0-x_0)\right\}^{\frac{1}{2}}.$$

14801. (R. F. MUIRHEAD, M.A., B.Sc.)—Show that, if the sum of any number of quantities = 0, then the sum of their cubes is three times the sum of their products taken three at a time.

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Solutions (I.) *by* ALETROP; (II.) *by* H. W. CURJEL, M.A., *and* LIONEL E. REAY, B.A.; (III.) *by* Professor SANJÁNA, M.A., *and others*; (IV.) *by* V. DANIEL, B.Sc.

(I.) Soit $x^m + A_1x^{m-1} + A_2x^{m-2} + \ldots + A_m = 0$ l'équation qui a pour racines les m quantités $a_1, a_2, \ldots, a_m$, dont la somme $S_1 = 0$. En égard de cette valeur dans les relations connues $S_1 = -A_1$, $S_3 = -A_1^3 + 3A_1A_2 - 3A_3$, il vient $S_3 = -3A_3$; d'ou la proposition.

(II.) $$\Sigma a^3 - 3\Sigma abc = \Sigma a(\Sigma a^2 - \Sigma ab) = 0.$$

Hence $$\Sigma a^3 = 3\Sigma abc.$$

(III.) Let the theorem be true for the k quantities $a_1, a_2, a_3, \ldots, a_k$; so that, $a_1 + a_2 + a_3 \ldots + a_k$ being zero, we have

$$a_1^3 + a_2^3 + a_3^3 \ldots + a_k^3 = 3\Sigma(a_1a_2a_3 \ldots a_k)_3.$$

Now let $a_1 = a_1 + a_2$; then

$$\begin{aligned} a_1^3 + a_2^3 + a_3^3 + a_3^3 \ldots + a_k^3 &= 3\Sigma[(a_1 + a_2)a_2a_3 \ldots a_k]_3 - 3a_1a_2(a_1 + a_2) \\ &= 3\Sigma[(a_1 + a_2)a_2a_3 \ldots a_k]_3 + 3a_1a_2(a_2 + a_3 \ldots + a_k) \\ &= 3\Sigma(a_1a_2a_3a_3 \ldots a_k)_3. \end{aligned}$$

But the theorem is true for three quantities; hence, it is true for 4, 5, ..., or any number of them.

(IV.) Let $a, b, c, d, \ldots$ be any number of quantities, and x an infinitesimal in comparison with the least. We have the identity

$$\{(1 + ax)(1 + bx)(1 + cx)(1 + dx) \ldots\} = 1 + x\Sigma a + x^2\Sigma ab + x^3\Sigma abc + \ldots,$$

which, when $\Sigma a = 0$, becomes

$$\{(1 + ax)(1 + bx)(1 + cx)(1 + dx) \ldots\} = 1 + x^2\Sigma ab + x^3\Sigma abc + \ldots.$$

Taking logarithms of both sides, and expanding, the coefficient of x^3 on the left is $\frac{1}{3}\Sigma a^3$, and on the right is Σabc. Hence, with the given condition,

$$\Sigma a^3 = 3\Sigma abc.$$

14820. (Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Find the product of the two series

$$0 + 24 + 216 + 960 + 3000 + \ldots, \quad \text{to } n \text{ terms},$$
$$2 + 40 + 270 + 1088 + 3250 + \ldots, \quad \text{to } n \text{ terms}.$$

Solutions (I.) *by* Rev. C. V. DURELL, M.A., Rev. T. ROACH, M.A., *and others*; (II.) *by* JOHN ORCHARD, M.A.

(I.) $S_1 = 0 + 24 + 216 + 960 + 3000 + \ldots$ to n terms

$= (1^5 - 1^3) + (2^5 - 2^3) + (3^5 - 3^3) + \ldots$ to n terms $= \Sigma n^5 - \Sigma n^3$,

$S_2 = 2 + 40 + 270 + 1088 + 3250 + \ldots$ to n terms

$= (1^5 + 1^3) + (2^5 + 2^3) + (3^5 + 3^3) + \ldots$ to n terms $= \Sigma n^5 + \Sigma n^3$;

therefore $$S_1S_2 = (\Sigma n^5)^2 - (\Sigma n^3)^2,$$

$\Sigma n^3 = \frac{1}{4}\{n^2(n+1)^2\}$ and $\Sigma n^5 = \frac{1}{12}\{n^2(n+1)^2\}(2n^2 + 2n - 1)$;

therefore $$\begin{aligned} S_1S_2 &= \tfrac{1}{144}\{n^4(n+1)^4\}\{(2n^2 + 2n - 1)^2 - 9\} \\ &= \tfrac{1}{36}\{n^4(n+1)^4\}(n^2 + n + 1)(n^2 + n - 2) \\ &= \tfrac{1}{36}\{n^4(n+1)^4(n+2)(n^2 - 1)\}. \end{aligned}$$

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(II.) $S_1 = 0 + 24 + 216 + 960 + 3000 + \ldots$ to n terms,

$S_2 = 2 + 40 + 270 + 1088 + 3250 + \ldots$ to n terms;

$$\tfrac{1}{2}(S_2 - S_1) = \sum_1^n n^3, \qquad \tfrac{1}{2}(S_1 + S_2) = \sum_1^n n^5;$$

therefore $$S_1 S_2 = \left[\sum_1^n n^5\right]^2 - \left[\sum_1^n n^3\right]^2 = \ldots.$$

Note on Two Associated "Riders."

By R. Chartres.

To find a point P within a triangle ABC, so that

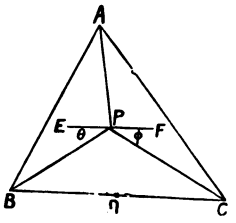
(1) $\Sigma(PA)$ = a minimum, (2) $\Sigma(PA)^2$ = a minimum.

Let P be the point; draw EPF perpendicular to PA, and let D be the middle of BC. Then PA is evidently a minimum to EF, and (1) the minimum PB + PC requires

$$\theta = \phi;$$

therefore APB = APC, ...;

(2) the minimum $PB^2 + PC^2$ requires DP to be a minimum, or DP to be perpendicular to EF; therefore APD is a median, &c.



15045. (E. V. Huntington, Ph.D.)—The centre of gravity of the equally weighted vertices and ortho-centre of any triangle is the centre of the nine-point circle of that triangle.

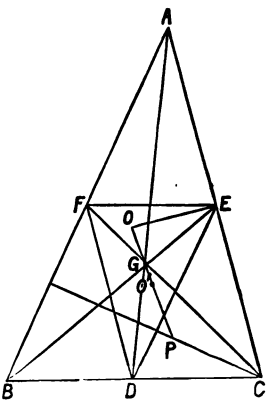
Solution by J. H. Taylor.

A proof of this problem follows readily from the fact that the ortho-centre and the centroid of a triangle are the centres of direct and inverse similitude of the circumscribed circle and the nine-point circle of the triangle. (*Pitt Press Euclid*, pp. 448, 449.)

For, if ABC be a triangle, D, E, F the middle points of the sides, G the centroid, P the ortho-centre, O the centre of the circle ABC, O′ the centre of the circle DEF, then OG = 2O′G

and OP = 2O′P;

whence 3GO′ = O′P.



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14947. (R. Tucker, M.A.)—PQ, a chord of a parabola, passes through a fixed point O, on the axis: the normals at P, Q intersect in R. Show that the locus of R, for varying positions of PQ, is a parabola, and also that the envelope of these parabolas, for varying positions of O, is the evolute of the primitive parabola.

Solutions (I.) *by the* Proposer, A. M. Nesbitt, M.A., *and many others;* (II.) *by* R. Knowles.

(I.) The equation to PQ is $y(m+m')-2x = 2amm'$; therefore, if AO = h, $h = -amm'$ .....(i.).

Normals at P, Q are

$$mx+y = am^3+2am, \quad m'x+y = am'^3+2am';$$

hence coordinates of R are $X = a(m^2+mm'+m'^2+2)$,

i.e., $X-2a+h = a(m^2+m'^2)$ and $y = h(m+m')$;

therefore locus of R is $ay^2 = h^2(x-2a-h)$.

The envelope is formed by eliminating h between

$$h^3+h^2(2a-x)+ay^2 = 0 \quad \text{and} \quad 3h = 2(x-2a),$$

i.e., it is $27ay^2 = 4(x-2a)^3$.

(II.) Let $y^2 = 4ax$ be the equation of the parabola; h, k coordinates of the pole of PQ; m, 0 those of O; then $h = -m$, and the coordinates of R are $x = 2a+k^2/a+m$, $y = mk/a$ .....(1, 2).

Eliminating k from (1), (2), the locus of R is the parabola

$$u \equiv ay^2-m^2x+2am^2+m^3 = 0 \text{ .....(3).}$$

From $du/dm = 0$, m is found $= \frac{3}{2}(x-2a)$, and, substituting in (3), the equation to the envelope is $27ay^2 = 4(x-2a)^3$, the equation to the evolute of the primitive parabola.

Proof of Euclid Book XI., Prop. 4.

By the late Professor R. B. Clayton.

Let OC be perpendicular to OA, OB, and let OP be any line through O in the plane AOB; then OC is perpendicular to OP. Through P draw APB perpendicular to OP. Then

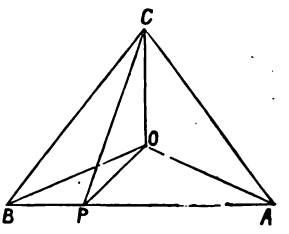
$AC^2-BC^2-AO^2-BO^2 = AP^2-BP^2$;

therefore CP is perpendicular to AB;

$\therefore\ CP^2+BP^2 = CB^2 = CO^2+BO^2$;

$\therefore\ CP^2-CO^2 = BO^2-BP^2 = PO^2$;

therefore COP is a right angle.



8796. (Professor Byomakesa Chakravarti, M.A.)—If OP, OQ be the arcs of a curve and of its circle of curvature intercepted between a point O and a neighbouring subtense perpendicular to the tangent, prove

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that OQ−OP is ultimately equal to $OQ^4/8\rho^3 . d\rho/ds$, s being the arc of the curve measured from a fixed point up to O and ρ the radius of curvature at O.

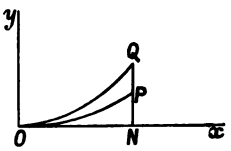
Solutions (I.) *by* Professor NANSON; (II.) *by* A. M. NESBITT, M.A.

(I.) Differentiating the equation $ds/dx = \sec\psi$ three times, we find, when $\psi = 0$, $ds/dx = 1$, $d^2s/dx^2 = 0$, $d^3s/dx^3 = 1/\rho^2$, $d^4s/dx^4 = -3/\rho^3 . d\rho/ds$.

Hence $OP = x + \frac{1}{6}(x^3/\rho^2) - \frac{1}{8}(d\rho/ds)\, x^4/\rho^3$, $OQ = x + \frac{1}{6}(x^3/\rho^2)$;

and therefore $OQ - OP = \frac{1}{8}(d\rho/ds)\, x^4/\rho^2$, $= \frac{1}{8}(OQ^4/\rho^3)\, d\rho/ds$ to fourth order.

(II.) Since s is a function of x, we may expand s by MACLAURIN'S theorem:



$$s = s_0 + s_0' x + s_0'' x^2/2! + \ldots,$$

where accents denote differentiation for x, and s_0', s_0'', ... are the values of s', s'', ..., when $x = 0$. Now

$$s' = \sec\psi \quad \text{and} \quad \psi' = 1/\rho . s';$$

therefore $$s'' = \sec^2\psi \tan\psi . 1/\rho,$$

$$s''' = 2\sec^3\psi \tan^2\psi . 1/\rho^2 + \sec^5\psi . 1/\rho^2 - (\sec^2\psi \tan\psi/\rho^2)\, \rho'.$$

These all hold for the circle except that ρ' is here zero, and the term in which it occurs vanishes when $x = 0$; so that we must proceed to a fourth differentiation. The only term in s_0'''' which will not disappear will arise from differentiating $1/\rho^2$ in the second term and $\tan\psi$ in the last term of s'''. Thus

$$s_0'''' = -[3\sec^5\psi/\rho^3 . \rho']_0 = -(3/\rho^3)(d\rho/dx),$$

which in the circle will vanish. Hence

$$OQ - OP = \frac{x^4}{4!}\frac{3}{\rho^3}\frac{d\rho}{dx} = \frac{OQ^4}{8\rho^3}\frac{d\rho}{ds} \text{ ultimately.}$$

15061. (J. PRESCOTT.)—From the fact that the inverse of a plane is a sphere through the centre of inversion, deduce that the inverse of a circle, with respect to a point not in its plane, is a circle.

Another Solution by R. F. WHITEHEAD.

The inverse of the meet of a sphere with a plane [or indeed of two spheres] is the meet of the two inverse spheres, and is therefore a circle.

5530. (J. W. L. GLAISHER, F.R.S.)—There being $(n+1)^2$ quantities

$$\begin{array}{cccccc} a_1a_1, & a_1a_2, & \ldots, & a_1a_n, & a_1a_{n+1}, \\ a_2a_1, & a_2a_2, & \ldots, & a_2a_n, & a_2a_{n+1}, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_na_1, & a_na_2, & \ldots, & a_na_n, & a_na_{n+1}, \\ a_{n+1}a_1, & a_{n+1}a_2, & \ldots, & a_{n+1}a_n, & a_{n+1}a_{n+1}, \end{array}$$

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of which only $\frac{1}{2}(n+1)(n+2)$ are different, as it is supposed that $a_p a_q = a_q a_p$, thence prove that

$$\begin{vmatrix} a_1a_1, & a_1a_2, & \dots, & a_1a_n, & a_1a_{n+1} \\ a_2a_1, & a_2a_2, & \dots, & a_2a_n, & a_2a_{n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_na_1, & a_na_2, & \dots, & a_na_n, & a_na_{n+1} \\ a_{n+1}a_1, & a_{n+1}a_2, & \dots, & a_{n+1}a_n, & a_{n+1}a_{n+1} \end{vmatrix} + \begin{vmatrix} a_1a_1, & a_1a_2, & \dots, & a_1a_n \\ a_2a_1, & a_2a_2, & \dots, & a_2a_n \\ \dots & \dots & \dots & \dots \\ a_na_1, & a_na_2, & \dots, & a_na_n \end{vmatrix}$$

$$= a_{n+1}a_{n+1} - \frac{(a_1a_{n+1})^2}{a_1a_1} - \frac{(a_2a_{n+1}\,.\,1)^2}{a_2a_2\,.\,1} - \frac{(a_3a_{n+1}\,.\,2)^2}{a_3a_3\,.\,2} - \dots - \frac{[a_na_{n+1}\,(n-1)]^2}{a_na_n\,(n-1)},$$

where $a_2a_{n+1}\,.\,1 = a_2a_{n+1} - \frac{a_2a_1\,.\,a_{n+1}a_n}{a_1a_1}$, $a_2a_2\,.\,1 = a_2a_2 - \frac{a_2a_1\,.\,a_2a_1}{a_1a_1}$;

and, generally, $a_pa_q\,.\,i = a_pa_q\,.\,i-1 - \frac{[a_pa_i\,(i-1)][a_qa_i\,(i-1)]}{a_ia_i\,(i-1)}$.

Solution by Professor NANSON.

The theorem is that, if, p, q having the values 1, 2, 3, ..., the quantities 2_{pq}, 3_{pq}, 4_{pq}, ... are derived from the quantities 1_{pq}, where $1_{pq} = 1_{qp}$ by the formula

$$r_{rr}\,(r+1)_{pq} = r_{rr}\,r_{pq} - r_{rp}\,r_{rq},$$

then the series $1_{nn} - 1^2_{1n}/1_{11} - 2^2_{2n}/2_{22} - 3^2_{3n}/3_{33} - \dots - r^2_{rn}/r_{rr}$

is the quotient obtained by dividing the symmetrical determinant $\begin{pmatrix} 1\,2\,\dots\,r\,n \\ 1\,2\,\dots\,r\,n \end{pmatrix}$ by its minor $\begin{pmatrix} 1\,2\,\dots\,r \\ 1\,2\,\dots\,r \end{pmatrix}$, the elements of the determinant being the quantities 1_{pq}.

To prove this we have first the formula

$$\begin{pmatrix} 1\,2\,\dots\,r \\ 1\,2\,\dots\,r \end{pmatrix}(r+1)_{pq} = \begin{pmatrix} 1\,2\,\dots\,r\,p \\ 1\,2\,\dots\,r\,q \end{pmatrix}.$$

For this formula is obviously true when $r = 1$, and, assuming it true for any value of r, we have by the law of formation

$$\begin{pmatrix} 1\,2\,\dots\,r+1 \\ 1\,2\,\dots\,r+1 \end{pmatrix}(r+2)_{pq}$$

$$= \left[\begin{pmatrix} 1\,2\,\dots\,r+1 \\ 1\,2\,\dots\,r+1 \end{pmatrix}\begin{pmatrix} 1\,2\,\dots\,r\,p \\ 1\,2\,\dots\,r\,q \end{pmatrix} - \begin{pmatrix} 1\,2\,\dots\,r+1 \\ 1\,2\,\dots\,r\,p \end{pmatrix}\begin{pmatrix} 1\,2\,\dots\,r+1 \\ 1\,2\,\dots\,r\,q \end{pmatrix}\right] \div \begin{pmatrix} 1\,2\,\dots\,r \\ 1\,2\,\dots\,r \end{pmatrix}$$

$$= \begin{pmatrix} 1\,2\,\dots\,r+1\,p \\ 1\,2\,\dots\,r+1\,q \end{pmatrix},$$

by JACOBI'S theorem relating to a minor of the adjugate, so that the formula is true for the next higher value of r, and therefore true always.

Next, the theorem stated is obviously true when $r = 1$. To prove it true in general then it is sufficient to prove that

$$\frac{\begin{pmatrix} 1\,2\,\dots\,r\,n \\ 1\,2\,\dots\,r\,n \end{pmatrix}}{\begin{pmatrix} 1\,2\,\dots\,r \\ 1\,2\,\dots\,r \end{pmatrix}} = \frac{\begin{pmatrix} 1\,2\,\dots\,r-1\,n \\ 1\,2\,\dots\,r-1\,n \end{pmatrix}}{\begin{pmatrix} 1\,2\,\dots\,r-1 \\ 1\,2\,\dots\,r-1 \end{pmatrix}} - \frac{r^2_{rn}}{r_{rr}},$$

i.e., that

$$\begin{pmatrix} 1\,2\,\dots\,r\,n \\ 1\,2\,\dots\,r\,n \end{pmatrix} = \left\{\begin{pmatrix} 1\,2\,\dots\,r-1\,n \\ 1\,2\,\dots\,r-1\,n \end{pmatrix}\begin{pmatrix} 1\,2\,\dots\,r \\ 1\,2\,\dots\,r \end{pmatrix} - \begin{pmatrix} 1\,2\,\dots\,r-1\,r \\ 1\,2\,\dots\,r-1\,n \end{pmatrix}^2\right\} + \begin{pmatrix} 1\,2\,\dots\,r-1 \\ 1\,2\,\dots\,r-1 \end{pmatrix},$$

and this is true by JACOBI'S theorem.

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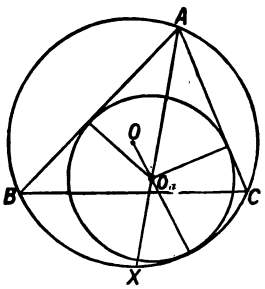
14913. (R. Tucker, M.A.) — O_a is the centre of the circle touching the circum-circle internally and also the sides AB, AC of the triangle ABC. Prove that

(1) $\rho_a \cos^2 \frac{1}{2}A = \rho_b \cos^2 \frac{1}{2}B = \rho_c \cos^2 \frac{1}{2}C = r$,

(2) $\Sigma OO_a = 3R - r\{1 + (\Sigma \tan \frac{1}{2}A)^2\}$,

where ρ_a, ρ_b, ρ_c are the respective radii of the touching circles.



Solution by Rev. T. Wiggins, B.A.

(1) $OO_a = R - \rho$; $O_aAC = \frac{1}{2}A$; $AE = \frac{1}{2}b$.

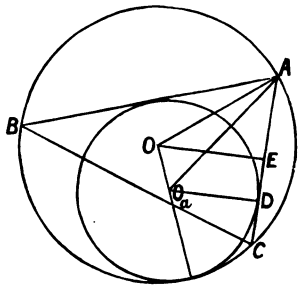
$\cos OAO_a = \cos(OAE - O_aAC)$

$= \cos(OAE - \frac{1}{2}A)$

$= b/2R \cos \frac{1}{2}A + (R^2 - \frac{1}{4}b^2)^{\frac{1}{2}} R/\sin \frac{1}{2}A$

$= b/2R (\cos \frac{1}{2}A + \cot B \sin \frac{1}{2}A)$

$= \sin B \cos \frac{1}{2}A + \cos B \sin \frac{1}{2}A$ ... (i.).



Now $(R-\rho)^2 = OO_a^2$

$= AO_a^2 + AO^2 - 2AO \cdot AO_a \cos OAO_a$;

$\therefore R^2 + \rho^2 - 2R\rho$

$= \rho^2 \operatorname{cosec}^2 \frac{1}{2}A$

$+ R^2 - 2R\rho \operatorname{cosec} \frac{1}{2}A \cos OAO_a$;

$\therefore \rho^2(\operatorname{cosec}^2 \frac{1}{2}A - 1)$

$= 2R\rho \operatorname{cosec} \frac{1}{2}A \cos OAO_a - 2R\rho$;

$\therefore \rho \cos^2 \frac{1}{2}A = 2R(\sin \frac{1}{2}A \cos OAO_a - \sin^2 \frac{1}{2}A)$

$= 4R(\sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C)$ on reduction from (i.)

$= 4R/abc\,(s-a)(s-b)(s-c) = (s-a)(s-b)(s-c)/S = S/s = r$.

(2) $\Sigma OO_a = \Sigma(R - \rho_a) = 3R - r\Sigma(\sec^2 \frac{1}{2}A) = 3R - r\{3 + \Sigma(\tan^2 \frac{1}{2}A)\}$

$= 3R - r\{1 + 2\tan \frac{1}{2}A \tan \frac{1}{2}B + 2\tan \frac{1}{2}B \tan \frac{1}{2}C$

$+ 2\tan \frac{1}{2}C \tan \frac{1}{2}A + \Sigma \tan^2 \frac{1}{2}A\}$

$= 3R - r\{1 + (\Sigma \tan \frac{1}{2}A)^2\}$.

13029. (Vve. F. Prime.)—Une circonférence, menée par le point de rebroussement d'une développée de parabole, coupe la courbe en quatre autres points que l'on joint au point de rebroussement. (1) Trouver le lieu du centre de la circonférence pour que le faisceau de ces quatre droites soit harmonique, et (2) démontrer que le centre des moyennes distances des quatre points d'intersection décrit une perpendiculaire à la tangente de rebroussement.

Solution by W. J. Greenstreet, M.A.

Let (at^2, at^3) be any point on the evolute, and let the equation of the circle be $x^2 + y^2 - 2gx - 2fy = 0$. Then the inclinations of the four joins

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to the axes are connected by the relations $at^4 + at^2 - 2ft - 2g = 0$, and the lines form an harmonic pencil if

$$\begin{vmatrix} a & 0 & \frac{1}{6}a \\ 0 & \frac{1}{6}a & \frac{1}{2}f \\ \frac{1}{6}a & \frac{1}{2}f & -2g \end{vmatrix} = 0,$$

and the centre (f, g) has for its locus $54y^2 + 72ax + a^2 = 0$. Also the centre of mean distances (X, Y) is given by

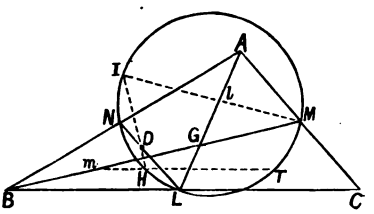
$$X = \tfrac{1}{4}a\Sigma t^2 = \tfrac{1}{4}a[(\Sigma t)^2 - 2\Sigma t_1 t_2] = -\tfrac{1}{2}a;$$

therefore it lies on a perpendicular to the tangent at the cusp.

14542. (Professor E. Lemoine.)—Dans un triangle ABC soient G le barycentre; L, M, N les milieux de BC, CA, AB; l, m les milieux de AG, BG; D le point situé sur LN au tiers de LN à partir de N. Pour trouver le centre de l'hyperbole de Kiepert, il suffira de prendre le point I où lM recoupe le cercle des neuf points; de prendre le point H où ID recoupe ce cercle; Hm recoupera le même cercle au centre T cherché.

Solution by C. E. Youngman, M.A.

Since the Kiepert hyperbola is rectangular and passes through ABCG, the point T ought to lie on the nine-point circles of ABC, ABG, &c.; *i.e.*, on the circles LMN, lmN, &c. Now, from the similar triangles DHL, DNI and DHN, DLI, we have



$$LH : HD = IN : ND$$

and $HD : HN = LD : LI$;

$$\therefore LH : HN = IN.LD : ND.LI = 2IN : LI.$$

But, from the similar triangles ILN, BCN (in which $\angle ILN = IMN = MNC = BCN$, and $\angle NIL = NML = NBC$), $IN : LI = BN : BC$;

therefore $LH : HN = BA : BC = ML : LB$.

Therefore, since $\angle LHN = MLB$, $\angle HLN = LMB = MBN = mlN$;

therefore $\angle mTN = mlN$, and T is concyclic with l, m, N.

14993. (Lt.-Col. Allan Cunningham, R.E.)—Let $\sigma(N)$ denote the *sum of the divisors* of a number N (including 1, but not N itself). Given $\sigma(N)$, to find N. Is this *always possible?* When is the solution *unique*, and when are there *multiple* solutions? Give examples.

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Solution by the PROPOSER.

The process of finding N from a *given* value of σN, being an *indirect* process, requires first a study of the effect of the direct process σ upon a given N. Here p, q denote *different* primes. The principal (simple) results are—

(1)	(2)	(3)	(4)	(5)	(6)	(7)
$N = p,$	$p^2,$	$pq,$	$2p,$	$4p,$	$8p,$	$2^n,$
$\sigma N = 1,$	$p+1,$	$p+q+1,$	$p+3,$	$3p+7,$	$7p+15,$	$2^n-1.$

Hence, when σN has any of the forms quoted, N has the form printed just above. One of the most interesting is—

Given $\sigma N = \omega$ (an *odd* number); then $\sigma N - 1$ is *even*.

Now, by GOLDBACH'S theorem, every *even* number is expressible as the *sum of two primes* (which may be equal); and when the even number is > 6, it appears that there will always be at least one solution $p + q = \sigma N - 1$, wherein p, q are *unequal* primes, both > 1; hence, when σN is an *odd* number > 7, formula (3) gives *at least one* solution $N = pq$, and (probably) always gives *more than one* solution when $\sigma N > 39$.

Ex.—$\sigma N =$	9;	11;	13;	15;	17;	... 39.
$N =$	3.5;	3.7;	5.7;	3.11;	3.13, 5.11;	... 7.31.

When σN is a given *even* number, N may often be found by one of the formulæ (2), (3), (6), or by similar more general formulæ, which can easily be devised; but the number of such formulæ required would be very great to attain any completeness.

When $\sigma N = 2, 5,$ and probably also when $\sigma N = 52, 88, 120,$ N is *impossible*. When σN is *small*, the solution is sometimes *unique*; *e.g.*,

$\sigma N =$	3,	4,	7,	9,	10,	11,	12,	28;
$N =$	4,	9,	8,	3.5,	2.7,	3.7,	11^2,	28.

Multiple solutions exist when σN has any *two* (or more) of the suitable forms above; *e.g.*, $\sigma N = 6$ gives $N = 5^2, 6$; $\sigma N = 8$ gives $N = 7^2, 2{\cdot}5$.

15064. (H. MACCOLL, B.A.)—Let ϕ denote the alternative statement $xyza' + xyb' + xy'z' + y'z'a'$. Express the implication $\epsilon : \phi$ (which asserts that ϕ is *certain*, *i.e.*, follows necessarily from our data) in the form $(M : z : N)(P : y : Q)(R : x : S)(T : \eta)$; in which M denotes the weakest statement from which (in conjunction with our data) we can deduce z, and N denotes the strongest conclusion we can draw from z. Similarly P, Q, R, S are to be interpreted. It is understood that M and N may contain y and x, but not z; that P and Q may contain x, but neither y nor z; and that R, S, T contain neither x nor y nor z, though they may contain a or b. The implication $T : \eta$ asserts that T is *impossible*, *i.e.*, inconsistent with our data and definitions. The symbol A′ is the denial of A, so that, whatever be the statement A, we have always $\epsilon : A + A'$ and $AA' : \eta$. Logicians will notice the connexion between this Question and the "inverse problem" of Prof. JEVONS; and mathe-

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maticians will notice the analogy between it and the problem to find the successive limits of integration in multiple integrals.

Solution by the PROPOSER.

It will be simpler to give a general solution for all problems of this type, and then apply the process to the above question as a particular case.

By the formulæ $\alpha : \beta = (\alpha\beta')'$ and $(\alpha : x)(\beta : x) = \alpha + \beta : x$, any given implication or implications may be reduced to a single implication of the form $As + Bs' + C : \eta$, in which the antecedent $As + Bs' + C$ is understood to be expressed in its simplest (or "primitive") form (see my Third Paper, *Proc. Lond. Math. Soc.*). This implication is evidently equivalent to $(B : s : A')(C : \eta)$; so that the required limits of s are B and A'. The result, after the elimination of s, is $(B : A')(C : \eta)$, which is equivalent to $AB + C : \eta$.

To find the limits of y from $AB + C : \eta$, we reduce $AB + C$ to its simplest form, which we will suppose to be $Dy + Ey' + F$. The implication $Dy + Ey' + F : \eta$ is equivalent to $(E : y : D')(F : \eta)$; so that the required limits of y are E and D'. The result, after the elimination of y, is $(E : D')(F : \eta)$, which is equivalent to $DE + F : \eta$.

To find the limits of x, we proceed exactly as before. We reduce $DE + F$ to its simplest form, which we will suppose to be $Gx + Hx' + K$; so that $DE + F : \eta$ takes the form $(H : x : G')(K : \eta)$, and the required limits of x are H and G'. The statements x, y, s having thus been successively eliminated, there remains the implication $GH + K : \eta$, which indicates the relation (if any) connecting the remaining statements after this elimination.

Since $\eta : \alpha$ and $\alpha : \epsilon$ are *certainties*, whatever the statement α may be, we can supply η for any missing antecedent, and ϵ for any missing consequent. For example, if, in seeking the limits of y and x, we find $(y : \alpha)(\beta : x)$, this result may be put into the form $(\eta : y : \alpha)(\beta : x : \epsilon)$.

In the given problem 15064 we have

$$\epsilon : \phi = \phi' : \eta = As + Bs' + C : \eta,$$

in which $A = y' + ab$, $B = by$, $C = x'y + ax'$.

From $AB + C : \eta$, we get $(y : a'x + b'x)(a : x)$; so that our final result is

$$(by : s : a'y + b'y)(\eta : y : a'x + b'x)(a : x : \epsilon)(\eta : \eta).$$

Hence, the required values of M, N, P, Q, R, S, T are $M = by$, $N = a'y + b'y$, $P = \eta$, $Q = a'x + b'x$, $R = a$, $S = \epsilon$, $T = \eta$. The result $T = \eta$ indicates that no relation connecting a and b (with no mention of s, y, x) can be deduced from the data of the problem.

Note.—If we had not reduced $Ax + Bx' + C$ to its simplest form, we should have found, not $s : a'y + b'y$, but $s : x(a'y + b'y)$. From this it might be supposed that the strongest conclusion deducible from s (in conjunction with our data) was not N, but xN. But then we have

$$(\epsilon : \phi) : (s : N : y : Q : x) : (N : x) : (N = xN);$$

so that here N and xN are equivalent, and neither of them is stronger than the other. The symbol $\alpha > \beta$, which asserts that α is stronger than β, is equivalent to $(\alpha : \beta)(\beta : \alpha)'$; that is, it asserts that α implies β, but that β does *not* imply α.

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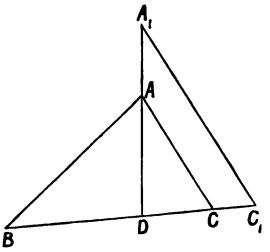
12471. (Professor SANJÁNA, M.A.)—Investigate the following (or similar) simple formulæ for the sides of a triangle of rational area :—

$$mx^2+my^2, \quad m^2+x^2y^2, \quad (x^2-m)(y^2+m).$$

Solution by the PROPOSER.

The area of ABC being rational, AD.BC is rational, *i.e.*, AD is rational; hence, AC being rational, if DC is not rational, it is an entire surd. Similarly, if BD is not rational, it is an entire surd; and the sum of these two surds is BC, a rational quantity, which is absurd. Thus the triangles ABD, ACD are required to be rational-sided right-angled triangles. Let DA_1C_1 have $A_1D = 2xy$, $DC_1 = x^2-y^2$, $C_1A = x^2+y^2$; then the sides of ADC are $2mxy$, $m(x^2-y^2)$, $m(x^2+y^2)$, where m is some constant multiplier. The sides of ABD may be represented by $2mxy$, $x^2y^2-m^2$, $x^2y^2+m^2$. Thus the sides of ABC are given by



$$mx^2+my^2, \quad mx^2-my^2-m^2+x^2y^2, \quad x^2y^2+m^2 \ldots\ldots\ldots\ldots\ldots(\alpha);$$

the area is seen to be $mxy(x^2-m)(y^2+m)$. Other similar expressions can be readily found. Let the sides A_1D, DC_1, C_1A_1 be respectively $2x$, x^2-1, x^2+1, as we may assume without loss of generality; change them in the ratio $\lambda : 1$, and let the new side $2x\lambda$ equal the side AD ($\equiv 2m$) of a triangle ADB, which has $DB = m^2-1$, $BA = m^2+1$. Then the sides of ABC are $\lambda(x^2+1)$, $\lambda(x^2-1)+m^2-1$, m^2+1, since $\lambda = m/x$. Substituting and multiplying out by x, we get

$$m(x^2+1), \quad (m+x)(mx-1), \quad x(m^2+1) \ldots\ldots\ldots\ldots\ldots(\beta).$$

Again, take $2x\lambda$ equal to AD ($\equiv m^2-1$), DB being now $2m$. Then $\lambda = (m^2-1)/2x$, and the sides of ABC are $\lambda(x^2+1)$, $\lambda(x^2-1)+2m$, m^2+1. Simplifying as before, we get

$$(m^2-1)(x^2+1), \quad (mx+1)^2-(m-x)^2, \quad 2x(m^2+1) \ldots\ldots\ldots\ldots(\gamma).$$

Lastly, take $A_1D = x^2-1$ in the triangle A_1DC_1; change the sides in the ratio $\lambda : 1$, and let the new side $\lambda(x^2-1)$ equal the side AD ($\equiv m^2-1$) of the triangle ADB. Then $\lambda = (m^2-1)/(x^2-1)$, and the sides of ABC are $\lambda(x^2+1)$, $2x\lambda+2m$, m^2+1. Proceeding as before, we get

$$(x^2+1)(m^2-1), \quad 2(m+x)(mx-1), \quad (m^2+1)(x^2-1) \ldots\ldots\ldots\ldots(\delta).$$

As particular cases we can find formulæ for the sides in terms of a single symbol. Thus let ADC have $AD = x^2-1$, $DC = 2x$, $CA = x^2+1$; place ADB contiguous, so that

$$AD = x^2-1, \quad DB = \left\{\tfrac{1}{2}(x^2-1)\right\}^2-1, \quad BA = \left\{\tfrac{1}{2}(x^2-1)\right\}^2+1.$$

Adding up and multiplying by 4, we get

$$4x^2+4, \quad x^4-2x^2+8x-3, \quad x^4-2x^2+5 \ldots\ldots\ldots\ldots\ldots(\epsilon).$$

The area is seen to be $2(x-1)(x+1)(x^2+2x-1)(x^2-2x+3)$. It appears that many other similar expressions could be obtained by means of various artifices.

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14097. (J. J. Barniville, B.A.)—Having $u_{n-1}+u_{n+1}=3u_n$, prove that

$$\frac{1}{10+11}+\frac{1}{13+11}+\frac{1}{29+11}+\frac{1}{74+11}+\ldots=\tfrac{3}{7}\frac{\sqrt{5}}{7+\sqrt{5}},$$

$$\frac{1}{2^2+1}+\frac{1}{3^2+1}+\frac{1}{7^2+1}+\frac{1}{18^2+1}+\ldots=\frac{\sqrt{5}+1}{10},$$

$$\frac{1}{1^2+9}-\frac{1}{4^2+9}+\frac{1}{11^2+9}-\frac{1}{29^2+9}+\ldots=\tfrac{1}{12},$$

$$\frac{1}{7+3}+\frac{1}{18-3}+\frac{1}{47+3}+\frac{1}{123-3}+\ldots=\tfrac{1}{8}.$$

Solution by Professor Sanjána, M.A.

As $10+11=3\cdot7$, $13+11=3\cdot8$, $29+11=5\cdot8$, $74+11=5\cdot17$, $193+11=12\cdot17$, ..., we get

$$\alpha=\frac{1}{3.7}+\frac{1}{3.8}+\frac{1}{5.8}+\frac{1}{5.17}+\ldots$$

$$=\frac{1}{3.7-}\ \frac{3^2.7^2}{3.7+3.8-}\ \frac{3^2.8^2}{3.8+5.8-}\ \frac{5^2.8^2}{5.8+5.17-}\ \ldots$$

$$=\frac{1}{3.7-}\ \frac{3^2.7^2}{3^2.5-}\ \frac{3^2.8^2}{8^2-}\ \frac{5^2.8^2}{5^2.5-}\ \frac{5^2.17^2}{17^2-}\ \ldots$$

$$=\frac{1}{3.7-}\ \frac{7^2}{5-}\ \frac{1}{1-}\ \frac{1}{5-}\ \frac{1}{1-}\ \ldots$$

$$=\frac{1}{21-49x},\qquad \text{where } x=\frac{1}{5-}\ \frac{1}{1-}\ \frac{1}{5-}\ \frac{1}{1-}\ \ldots=\frac{5-\sqrt{5}}{10}.$$

Thus the first series $=\dfrac{10}{-35+49\sqrt{5}}=\dfrac{2.5}{7\sqrt{5}(7-\sqrt{5})}=\tfrac{3}{7}\dfrac{\sqrt{5}}{7-\sqrt{5}}$.

Again, $\delta=\frac{1}{8}\left\{\frac{1}{3}+\frac{1}{3}+\frac{1}{10}+\frac{1}{24}+\ldots\right\}$

$$=\tfrac{1}{8}\left\{\frac{1}{1.2}+\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.8}+\ldots\right\}$$

$$=\tfrac{1}{8}\left\{\frac{2-1}{1.1.2}+\frac{3-1}{1.2.3}+\frac{5-2}{2.3.5}+\frac{8-3}{3.5.8}+\ldots\right\}$$

$$=\tfrac{1}{8}\left\{\frac{1}{1.1}-\frac{1}{1.2}+\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{2.3}-\frac{1}{3.5}+\frac{1}{3.5}-\ldots\right\}=\tfrac{1}{8}.$$

It will be found that

$$5\beta=\tfrac{1}{1}+\tfrac{1}{3}+\tfrac{1}{10}+\tfrac{1}{36}+\ldots$$

$$=\frac{1}{1-}\ \frac{1^2}{1+1.2-}\ \frac{2^2}{1.2+2.5-}\ \frac{10^2}{2.5+5.13-}\ \ldots$$

$$=\frac{1}{1-}\ \frac{1}{3-}\ \frac{1}{3-}\ \frac{1}{3-}\ \ldots=\frac{1}{1-\frac{1}{3}(3-\sqrt{5})}=\frac{2}{\sqrt{5}-1}=\frac{\sqrt{5}+1}{2}.$$

Lastly, $\gamma=\frac{1}{10}+\frac{1}{130}+\frac{1}{370}\ldots-(\frac{1}{26}+\frac{1}{260}+\frac{1}{962}\frac{1}{10}+\ldots)$;

hence $5\gamma=\dfrac{1}{1.2}+\dfrac{1}{2.13}+\dfrac{1}{13.89}\ldots-\left(\dfrac{1}{1.5}+\dfrac{1}{5.34}+\dfrac{1}{34.233}\ldots\right)$

$$=\frac{1}{2-}\ \frac{2^2}{2^2.7-}\ \frac{2^2.13^2}{13^2.7-}\ \ldots-\left(\frac{1}{5-}\ \frac{5^2}{5^2.7-}\ \frac{5^2.34^2}{34^2.7-}\ \ldots\right)$$

$$=\frac{1}{2-}\ \frac{1}{7-}\ \frac{1}{7-}\ \ldots-\frac{1}{5-}\ \frac{1}{7-}\ \frac{1}{7-}\ \ldots$$

$$=\frac{\sqrt{5}+1}{6}-\frac{\sqrt{5}-1}{6}=\tfrac{1}{3}.$$

[Series β and γ exemplify a general rule which I have already sent.]

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15070. (C. F. Sandberg, M.A.)—If, for all values of r from 0 to n,

$$a_r = a_r + ra_{r-1}x + \frac{r(r-1)}{1.2}a_{r-2}x^2 + \ldots + ra_1x^{r-1} + a_0x^r,$$

then, for the same values of r, $\quad a_r = a_r - ra_{r-1}x + \frac{r(r-1)}{1.2}a_{r-1}x^2 + \ldots + (-1)^r a_0 x^r.$

Solution by Constance I. Marks, B.A.

$$\begin{aligned}
a_r &= a_r + {}_rC_1 a_{r-1}x + {}^rC_2 a_{r-2}x^2 \quad + \ldots + {}^rC_s a_{r-s}x^s \quad + \ldots + {}^rC_{r-1}a_1x^{r-1} + a_0x^r,\\
-{}^rC_1x\,.\,a_{r-1} &= -{}^rC_1a_{r-1}x - {}^rC_1{}^{r-1}C_1a_{r-2}x^2 - \ldots - {}^rC_1{}^{r-1}C_{s-1}a_{r-s}x^s - \ldots - {}^rC_1{}^{r-1}C_{r-2}a_1x^{r-1} - {}^rC_1a_0x^r,\\
+{}^rC_2x^2\,.\,a_{r-2} &= \quad + {}^rC_2a_{r-2}x^2 \quad + \ldots + {}^rC_2{}^{r-2}C_{s-2}a_{r-s}x^s + \ldots + {}^rC_2{}^{r-2}C_{r-3}a_1x^{r-1} + {}^rC_2a_0x^r,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
+(-1)^s{}^rC_sx^sa_{r-s} &= \quad (-1)^s{}^rC_sa_{r-s}x^s \quad + \ldots + (-1)^s{}^rC_s{}^{r-s}C_{r-s-(r-s+1)}a_1x^{r-1} + (-1)^s{}^rC_sa_0x^r,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
+(-1)^r x^r a_0 &= \quad + (-1)^r a_0 x^r.
\end{aligned}$$

On the left is the required series. Consider the general coefficient on the right.

$$\begin{aligned}
\text{Coefficient } a_{r-s}x_s &= {}^rC_s - {}^rC_1{}^{r-1}C_{s-1} + {}^rC_2{}^{r-2}C_{s-2} - \ldots + (-1)^s{}^rC_s\\
&= {}^rC_s\left\{1 - \frac{s}{1!} + \frac{s(s-1)}{2!} - \frac{s(s-1)(s-2)}{3!} + \ldots + (-1)^s\right\} = {}^rC_s(1-1)^s = 0.
\end{aligned}$$

Therefore all coefficients on the right except this first vanish; therefore

$$a_r = a_r - ra_{r-1}x + \frac{r(r-1)}{2!}a_{r-2}x^2 - \ldots + (-1)^r a_0 x^r;$$

and the method shows that the result is true for the proposed range of values of r.

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15029. (Professor H. Langhorne Orchard, M.A., B.Sc.)—Show that the value of

$$\left[\int_0^\infty \left(\frac{x^4+9}{3x^3}\right) e^{-(x^4+9)/3x^3}\,dx\right] \Big/ \left[\int_0^\infty \left(\frac{x^4-9}{3x^3}\right) e^{-(x^4+9)/3x^3}\,dx\right]$$

lies between 4 and 6.

Solution by θ. β.

Write $2y = (x^4+9)/3x^3$; then, from $x = 0$ to $\sqrt{3}$, we have

$$x = (\tfrac{1}{3})^{\frac{1}{2}}\left[(y+1)^{\frac{1}{2}} - (y-1)^{\frac{1}{2}}\right],$$

and, from $x = \sqrt{3}$ to ∞, we have

$$x = (\tfrac{1}{3})^{\frac{1}{2}}\left[(y+1)^{\frac{1}{2}} + (y-1)^{\frac{1}{2}}\right];$$

further, in the first range y varies from ∞ to 1, and in the second from 1 to ∞. Also in the first range we must put $(x^4-9)/3x^3 = -2(y^2-1)^{\frac{1}{2}}$, and in the second $(x^4-9)/3x^3 = +2(y^2-1)^{\frac{1}{2}}$. Then

$$\begin{aligned} \mathrm{N} = \int_0^\infty \frac{x^4+9}{3x^3} e^{-(x^4+9)/3x^3}\,dx &= (\tfrac{1}{3})^{\frac{1}{2}} \int_\infty^1 2ye^{-2y}\,\tfrac{1}{2}\left[\frac{1}{(y+1)^{\frac{1}{2}}} - \frac{1}{(y-1)^{\frac{1}{2}}}\right] dy \\ &\quad + (\tfrac{1}{3})^{\frac{1}{2}} \int_1^\infty 2ye^{-2y}\,\tfrac{1}{2}\left[\frac{1}{(y+1)^{\frac{1}{2}}} + \frac{1}{(y-1)^{\frac{1}{2}}}\right] dy \\ &= 6^{\frac{1}{2}} \int_1^\infty \frac{y\,dy}{(y-1)^{\frac{1}{2}}} e^{-2y}. \end{aligned}$$

Similarly

$$\begin{aligned} \mathrm{D} = \int_0^\infty \frac{x^4-9}{3x^3} e^{-(x^4+9)/3x^3}\,dx &= -\int_\infty^1 (\tfrac{1}{3})^{\frac{1}{2}}\, 2(y^2-1)^{\frac{1}{2}} e^{-2y}\,\tfrac{1}{2}\left[\frac{1}{(y+1)^{\frac{1}{2}}} - \frac{1}{(y-1)^{\frac{1}{2}}}\right] dy \\ &\quad + \int_1^\infty (\tfrac{1}{3})^{\frac{1}{2}}\, 2(y^2-1)^{\frac{1}{2}} e^{-2y}\,\tfrac{1}{2}\left[\frac{1}{(y+1)^{\frac{1}{2}}} + \frac{1}{(y-1)^{\frac{1}{2}}}\right] dy \\ &= 6^{\frac{1}{2}} \int_1^\infty (y-1)^{\frac{1}{2}} e^{-2y}\,dy. \end{aligned}$$

Finally, put $y = 1 + \frac{1}{2}t$ in each integral, and we get

$$\mathrm{N} = \tfrac{1}{2}\sqrt{3}e^{-2}\int_0^\infty (t^{\frac{1}{2}} + 2t^{-\frac{1}{2}})\, e_{-t}\,dt = \tfrac{1}{2}\sqrt{3}e^{-2}\left[\Gamma(\tfrac{3}{2}) + 2\Gamma(\tfrac{1}{2})\right] = \tfrac{5}{4}(3\pi)^{\frac{1}{2}} e^{-2},$$

$$\mathrm{D} = \tfrac{1}{2}\sqrt{3}e^{-2}\int_0^\infty t^{\frac{1}{2}} e^{-t}\,dt = \tfrac{1}{2}\sqrt{3}e^{-2}\,\Gamma(\tfrac{3}{2}) = \tfrac{1}{4}(3\pi)^{\frac{1}{2}} e^{-2}.$$

Thus N/D = 5 exactly.

This is a neat, if unnecessary, illustration of the method to be followed if, when making a substitution $y = f(x)$, the function $f(x)$ is stationary at a finite number of points in the range of variation of x. As a verification of the expression for N, we may use the theorem

$$\int_0^\infty f(ax + b/x)\,dx = 1/a \int_0^\infty f\left[\sqrt{(z^2 + 4ab)}\right] dz$$

(Stolz, *Grundzüge*, Theil I., p. 417), where a, b are both positive, and the integrals are supposed to have a meaning. But here we have

$$(x^4+9)/3x^3 = (x/\sqrt{3} + \sqrt{3}/x)^2 - 2\ ;$$

so that $\mathrm{N} = \sqrt{3}\int_0^\infty (z^2+2)\, e^{-(z^2+2)}\,dz = \sqrt{(3\pi)}\,(\tfrac{1}{2}+1)\, e^{-2} = \tfrac{5}{4}\sqrt{(3\pi)}\, e^{-2}$, as found before.

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9133. (W. J. GREENSTREET, M.A.)—Prove that there are

$$(p-1)\{(p^2+1)^n-1\}/p^2$$

integers less than $(p^2+1)^n$ which are divisible by p and not by p^2 (n and p integral).

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.; (II.) *by* R. CHARTRES, *and others.*

(I.) Let s' = number of numbers ($<N$) containing p,

s'' = " " ($<N$) " p^2,

$E(x)$ = integer part of the number x.

Then $s' = E[(N-1)/p]$, $s'' = E[(N-1)/p^2]$.

Let $N = (p^2+1)^n$; then every term of $(N-1)$ contains both p and p^2. Hence, in this case,

$$s' = [(p^2+1)^n-1]/p, \quad s'' = [(p^2+1)^n-1]/p^2,$$

and the *number sought* $= s'-s'' = (1/p-1/p^2)[(p^2+1)^n-1]$, which reduces to the form given.

(II.) $(p^2+1)^n-1 \equiv Ap^2$ numbers to be considered, of which the favourable cases are p, $2p$, ..., $(p+1)p$, $(p+2)p$, ... $= (p-1)/p^2$ of the above.

14917. (A. M. NESBITT, M.A.)—Three conics S_1, S_2, S_3 have a common point O. The remaining intersections of S_2S_3 are the vertices of a triangle Δ_1, those of S_3S_1 of a triangle Δ_2, and those of S_1S_2 of a triangle Δ_3. Prove that the nine sides of these three triangles all touch the same conic U. Or, reciprocally: Given three parabolas, their common tangents, taking the curves in pairs, will form three triangles (one for each pair) whose nine vertices lie on a conic.

Solutions (I.) *by* Professor MORLEY; (II.) *by* Professor NANSON; (III.) *by* Professor JAN DE VRIES; (IV.) *by the* PROPOSER.

(I.) Consider the more general theorem: the three common four-lines of three conics S_1, S_2, S_3, taken in pairs, are inscribed in a cubic curve. For let the triangle of reference be three lines each touching two conics. The conics are then

$$(1) \quad \xi_1^2 + l_1\xi_2\xi_3 + m_1\xi_3\xi_1 + n_1\xi_1\xi_2 = 0,$$

$$(2) \quad \xi_2^2 + l_2\xi_2\xi_3 + m_2\xi_3\xi_1 + n_2\xi_1\xi_2 = 0,$$

$$(3) \quad \xi_3^2 + l_3\xi_2\xi_3 + m_3\xi_3\xi_1 + n_3\xi_1\xi_2 = 0.$$

Eliminating ξ_1 from (2) and (3), we have

$$n_3\xi_2^3 - m_2\xi_3^3 + \text{terms in both } \xi_2, \xi_3 = 0.$$

Hence a cubic, whose cube terms are $l_1x_1^3 + m_2x_2^3 + n_3x_3^3$, can pass through those vertices of the three four-lines which lie on the triangle of reference. (A) That is, *these nine points are the basis-points of a pencil of cubics.*

Draw a cubic through three points of one four-line which are not on a line, and through three points of each of the other four-lines which are on a line. By (A), the cubic contains all points of the first four-line. Now take other collinear points of the second four-line; we have again

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a cubic meeting the former in ten points, therefore the same cubic. Thus the theorem is proved.

If now the conics have a common line, nine of the eighteen points are on it, and the cubic breaks up into a line and a conic through the other nine points, which is Mr. NESBITT's theorem. If the conics have *two* lines in common, we have in reciprocal form the theorem that the radical axes of three circles meet.

(II.) The envelope of the degenerate conics of the system $l_1S_1 + l_2S_2 + l_3S_3$ is a class cubic which when S_1, S_2, S_3 have a common point O degenerates into the point O and a conic touching the sides of Δ_1, Δ_2, Δ_3.

(III.) Let A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 denote the angular points Δ_1, Δ_2, Δ_3. The pencil of conics passing through C_1, C_2, C_3, and O determines on S_3 the triads of a cubic involution. As is known, the triangles formed by these triads are circumscribed to a conic U. To these triads belong A_1, A_2, A_3 and B_1, B_2, B_3. The degenerate conic formed by the lines OC_1 and C_2C_3 gives a triad of which one point D_1 is determined by OC_1 and two points D_2, D_3 by C_2C_3. Similarly the systems OC_2, C_3C_1 and OC_3, C_1C_2 intersect S_3 in the points E_2, E_3, E_1 and F_3, F_1, F_2. Now, U will be touched by the lines $D_2D_3 \equiv C_2C_3$, $E_3E_1 \equiv C_3C_1$, $F_1F_2 \equiv C_1C_2$, which form the triangle $C_1C_2C_3$; hence Δ_1, Δ_2, and Δ_3 are circumscribed to the same conic U.

(IV.) Take Δ_3 for triangle of reference ABC, and let the equations of the three conics be

$$S_1 \equiv p/x + q/y + r/z = 0, \quad S_2 \equiv p'/x + q'/y + r'/z = 0 \quad \ldots\ldots(1, 2),$$

$$S_3 \equiv (abcfgh)(xyz)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

So that O, the fourth intersection of S_1S_2, lies on S_3.

Now the conic $U \equiv \sqrt{(\lambda x)} + \sqrt{(\mu y)} + \sqrt{(\nu z)}$ is inscribed in Δ_3; and, provided the invariant relation $\Theta^2 = 4\Delta\Theta'$ subsists between U and S_3, we can certainly arrange that it shall also be inscribed in *either* Δ_1 *or* Δ_2. I propose to show that, if $\lambda : \mu : \nu :: a/P : b/Q : c/R$, where

$$P = \begin{vmatrix} q & r \\ q' & r' \end{vmatrix}, \quad Q = \begin{vmatrix} r & p \\ r' & p' \end{vmatrix}, \quad R = \begin{vmatrix} p & q \\ p' & q' \end{vmatrix},$$

U will, in fact, be inscribed *both* in Δ_1 *and* Δ_2.

Expressing that S_3 goes through O, the fourth intersection of S_1S_2, whose coordinates are (1/P, 1/Q, 1/R) or $(\lambda/a, \mu/b, \nu/c)$, we get

$$bc\lambda^2 + ca\mu^2 + ab\nu^2 + 2af\mu\nu + 2bg\nu\lambda + 2ch\lambda\mu = 0,$$

and this is identical with the invariant relation $\Theta^2 = 4\Delta\Theta'$ above referred to.

Now, if $(x_1y_1z_1)$ be a vertex A_1 of Δ_1, and $lx_1 + my_1 + nz_1 = 0$ be the equation to its opposite side B_1C_1, and if $L = 0$ be the equation of OA_1, the equation

$$KS_3 = L(l_1x + m_1y + n_1z) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

can be made to represent *any* conic through $OA_1B_1C_1$ by suitably choosing K, *i.e.*, the coefficients of x^2, y^2, z^2 in (4) can be made to vanish simultaneously, since the conic S_3 passes through $OA_1B_1C_1$. Now

$$L \equiv \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ \lambda/a & \mu/b & \nu/c \end{vmatrix}.$$

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Hence we may put

$$K\alpha = l_1(y_1\nu/c - z_1\mu/b), \quad K\beta = m_1(z_1\lambda/a - x_1\nu/c), \quad K\gamma = n_1(x_1\mu/b - y_1\lambda/a),$$

or
$$K\lambda/l_1 = (y_1Q - z_1R)/PQR,$$

with two similar equations; whence, by addition, $\lambda/l_1 + \mu/m_1 + \nu/n_1 = 0$; or B_1C_1 touches U. But B_1C_1 was *any* side of Δ_1, and therefore *all* the sides of Δ_1 touch U; and similarly *all* the sides of Δ_2 touch U.

It is remarkable that the equation of U does not in any way depend upon the values of fgh in the equation to S_3. Thus, if we take

$$S_4 \equiv (abcf'g'h')(xyz)^2 = 0$$

in place of S_3, we arrive at the *same* conic U as before, provided that S_4 passes through O. Also the conic $S_3 - S_4 = 0$, which passes through O and the other three intersections of S_3 and S_4, obviously passes also through ABC.

14765. (Professor Jan de Vries.)—The equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + b_1x + b_2y = 0$$

being referred to rectangular axes, the axes are supposed to turn about the origin. Prove that the function $I = (a_{11} - a_{22})\,b_1b_2 - a_{12}(b_1^2 - b_2^2)$ does not undergo change of form. What is the proper geometric meaning of $I = 0$?

Solution (I.) *by the* Proposer; (II.) *by* Constance I. Marks, B.A., *and* Professor Sanjána, M.A.

Using known symbols, we put

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = (a_1x_1 + a_2x_2)^2 = a_x^2.$$

The axes being turned through an angle ϕ, we have, denoting the new co-ordinates by ξ_1, ξ_2, $x_1 = \xi_1\cos\phi + \xi_2\sin\phi$, $x_2 = -\xi_1\sin\phi + \xi_2\cos\phi$.

Then
$$a_x = a_1x_1 + a_2x_2 = (a_1\cos\phi - a_2\sin\phi)\,\xi_1 + (a_1\sin\phi + a_2\cos\phi)\,\xi_2$$
$$= \alpha_1\xi_1 + \alpha_2\xi_2.$$

Hence $\alpha_1 = a_1\cos\phi - a_2\sin\phi$, $\alpha_2 = a_1\sin\phi + a_2\cos\phi$.

Since $\xi_1 = x_1\cos\phi - x_2\sin\phi$, $\xi_2 = x_1\sin\phi + x_2\cos\phi$,

the symbolical coefficients α_1, α_2 are transformed by the same substitution as the variables.

If b_1, b_2 denote a new set of symbolical coefficients, it is easily proved that the functions $a_1b_1 + a_2b_2$ and $a_1b_2 - a_2b_1$ are orthogonal invariants. Thence we get the invariant

$$I = (a_1b_1 + a_2b_2)(a_1b_2 - a_2b_1).$$

Or, writing a_{11}, a_{12}, and a_{22} instead of a_1^2, a_1a_2, and a_2^2,

$$I = (a_{11} - a_{22})\,b_1b_2 - a_{12}(b_1^2 - b_2^2).$$

In order to find the geometric meaning of $I = 0$, take the tangent of the conic at the origin as axis OY. Then the conic will be represented by an equation of the form $a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + \beta_1\xi = 0$, and $I = -a_{12}\beta_1^2 = 0$, or $a_{12} = 0$. Hence it follows that the origin is a vertex of the conic.

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(II.) Given equation is $a_{11}x^2+2a_{12}xy+a_{22}y^2+b_1x+b_2y=0$ ..... (i.).

Let the axes turn about the origin through an arbitrary angle θ to positions OX, OY, *i.e.*, let $x=X\cos\theta-Y\sin\theta$, $y=X\sin\theta+Y\cos\theta$.

Then (i.) becomes

$$a_{11}(X\cos\theta-y\sin\theta)^2+2a_{12}(X\cos\theta-Y\sin\theta)(X\sin\theta+Y\cos\theta)+\ldots=0;$$

that is,
$$\begin{aligned}X^2&(a_{11}\cos^2\theta+2a_{12}\sin\theta\cos\theta+a_{22}\sin^2\theta)\\&+2XY(-a_{11}\sin\theta\cos\theta+a_{12}\cos^2\theta-\sin^2\theta+a_{22}\sin\theta\cos\theta)\\&+Y^2(a_{11}\sin^2\theta-2a_{12}\sin\theta\cos\theta+a_{22}\cos^2\theta)\\&+X(b_1\cos\theta+b_2\sin\theta)\\&+Y(-b_1\sin\theta+b_2\cos\theta)=0.\end{aligned}$$

Hence, if given function in transformed equation be I′,

$$\begin{aligned}I'&=\left[\left\{(a_{11}-a_{22})\cos2\theta+2a_{12}\sin2\theta\right\}\left\{b_1b_2\cos2\theta-\tfrac{1}{2}(b_1^2-b_2^2)\sin2\theta\right\}\right.\\&\quad\left.-\left\{a_{12}\cos2\theta-\tfrac{1}{2}(a_{11}-a_{22})\sin2\theta\right\}\left\{(b_1^2-b_2^2)\cos2\theta+2b_1b_2\sin2\theta\right\}\right]\\&=\left[(a_{11}-a_{22})\,b_1b_2(\cos^2 2\theta+\sin^2 2\theta)-a_{12}(b_1^2-b_2^2)(\sin^2 2\theta+\cos^2 2\theta)\right]\\&=(a_{11}-a_{22})\,b_1b_2-a_{12}(b_1^2-b_2^2)\\&=I;\end{aligned}$$

therefore the function is an invariant of the conic (i.)

So long as the origin of rectangular axes is unchanged, the pair of lines through it parallel to the asymptotes, the lines bisecting the angles between these and the tangent at the origin to the given conic, are lines independent of the axes of reference.

These are represented in terms of the equation (i.) by

$$a_{11}x^2+2a_{12}xy+a_{22}y^2=0,\quad\frac{a_{11}-a_{22}}{a_{12}}=\frac{x^2-y^2}{xy},\quad b_1x+b_2y=0\ldots\text{(ii., iii., iv.)}.$$

Hence, if $I=0$, or $\dfrac{a_{11}-a_{22}}{a_{12}}=\dfrac{b_1^2-b_2^2}{b_1b_2}$,

the line represented by (iv.) coincides with one of the lines represented by (iii.), that is, the tangent to the conic at the chosen origin is equally inclined to the asymptotes. Geometrically speaking then, a vertex has been chosen as origin, and $I=0$ expresses the fact that the tangent at a vertex makes equal angles with the asymptotes.

14800. (R. F. Davis, M.A.)—Prove that the equations $xy=ac+bu$, $yz=bd+cv$, $zu=ce+dx$, $uv=da+ey$, $vx=eb+az$ are not independent, and that, given any three, the other two follow.

Solutions (I.) *by* H. W. Curjel, M.A., J. H. Taylor, M.A., *and others*; (II.) *by* Aletrop; (III.) *by* Professor Sanjána, M.A.

(I.) The symmetry of the equations shows that it is immaterial which three are considered as given. If the first three are given, then

$$cvx=xyz-xbd=acz+buz-xbd=acz+bce$$

and
$$cuv=uyz-bdu=cey+dxy-bdu=cey+cda;$$

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therefore $vx = eb + az$ and $uv = da + ey$, unless $c = 0$. If $c = 0$, then of the first three equations two only are independent, for they may be written $u/x = y/b = d/z$, and from these with one of the remaining two the remaining equation follows.

(II.) Démontrons, par exemple, que (4) et (5) sont conséquences de (1), (2) et (3). En divisant (2) par (3), on trouve $c(uv - ey) = d(xy - bu)$. En y introduisant la valeur de xy donné par (1), il vient $uv - ey = da$, qui n'est autre que (4). Maintenant, on déduit de même (5) de (2), (3) et (4). On pourrait aussi s'appuyer sur ce qu'on peut supposer que a, b, c, d, e soient les côtés d'un pentagone cyclique et x, y, z, u, v les diagonales joignant les sommets de deux en deux.

(III.) Let the first three be given

$$y \,.\, x - c \,.\, a - b \,.\, u = 0, \quad y \,.\, z - c \,.\, v - b \,.\, d = 0, \quad zu = ce + dx \ldots (1, 2, 3).$$

From (1) and (2) by cross multiplication

$$\frac{y}{ad - uv} = \frac{b}{-xv + az} = \frac{c}{-uz + dx} = \frac{c}{-ce} \text{ [by (3)]} = -\frac{1}{e}.$$

Hence $ye + ad = uv$ .....(4)

and $eb + az = vx$ .....(5).

10198. (John Griffiths, M.A.) — The coordinates x, y, z being proportional to the perpendicular distances from the sides of a triangle ABC, prove that (1) the locus of a point (x, y, z) such that its pedal circle (*i.e.*, the circle which passes through the feet of the perpendiculars drawn from the point in question upon the sides of the triangle) touches the nine-point circle is the cubic

$$x(y^2 - z^2)\cos A + y(z^2 - x^2)\cos B + z(x^2 - y^2)\cos C = 0;$$

and (2) trace this curve.

Solution by W. H. Blythe, M.A.

(1) It may be shown that the radical axis of the circum-circle with that of the pedal circle is

$$\lambda\alpha + \mu\beta + \nu\gamma = 0,$$

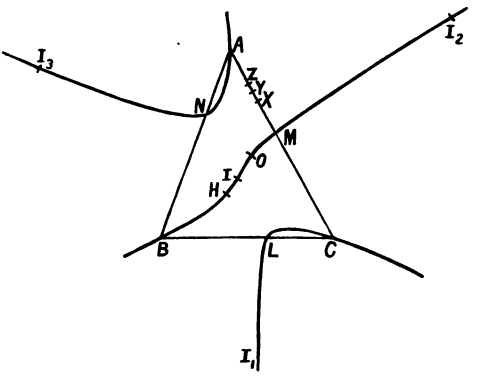
where

$$\lambda(pbc + qca + rab + a^2 + b^2 + c^2) = bc(1 + p\cos A + \cos^2 A),$$

with similar expressions for μ and ν, and where

$$pyz = y^2 + z^2,$$

with similar expressions for q and r.



(See Ferrers, *Trilinear Cons.*, pp. 85–93.

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Now the radical axis of the nine-points and circum-circle is

$$\tfrac{1}{2}(\alpha\cos A+\beta\cos B+\gamma\cos C)=0;$$

therefore the equation of the radical axis of the nine-points circle with the pedal circle is

$$2(\lambda\alpha+\mu\beta+\nu\gamma)=\alpha\cos A+\beta\cos B+\gamma\cos C.$$

Eliminate α between this equation and that of the nine-points circle, and express the condition that the resulting expression may be a complete square: we find

$$(p^2-4)\cos^2 A-2(qr-2p)\cos B\cos C+\text{similar terms}=0.$$

This then is the condition that the circles touch one another. Multiply by $x^2y^2z^2$ and substitute values of p, q, r: the result is the square of

$$x(y^2-z^2)\cos A+y(z^2-x^2)\cos B+z(x^2-y^2)\cos C=0.$$

(2) To trace the curve, we notice that x, y, z is the focus of a conic touching the sides of the triangle, the auxiliary circle of which touches the nine-points circle.

The curve passes through A, B, C, H, O, L, M, N, I, I_1, I_2, I_3, namely, the angular points of the triangle, the ortho-centre, circum-centre, the points at which AO, BO, CO meet the opposite sides, the in-centre, and four escribed centres. Also HA, HB, HC, HO are the tangents drawn from H, and OI, OI_1, OI_2, OI_3 the tangents from O.

This is sufficient to determine the curve, which is a cubic having three real asymptotes; one serpentine branch touches one asymptote and contains an angular point, the ortho-centre, in-centre, circum-centre, cuts the opposite side at a known point, and passes on to the opposite escribed centre.

When the curve cuts the circum-circle, which it does at three points besides A, B, C, the point of intersection is the focus of a parabola which touches the sides, and the direction of the axis of this parabola must give the direction of an asymptote; for the inverse point, which is also on the curve, is at infinity.

Suppose the angle A in the figure to move along AC, the positions of B and C being unchanged. If at Z the angle ZBC = ZCB, the curve degenerates into a straight line ZO and a hyperbola the transverse axis of which is ZO, and which passes through BNI_3CMI_2, HB and HC being tangents.

If A still continues to move along AC between Z and Y, we find that the serpentine branch passes through A. At Y, where the angle CAB = CBA, the curve again degenerates into a straight line CO and the corresponding hyperbola.

Between Y and X the serpentine passes through C till we obtain the final change, when the angle BAC = BCA, and the serpentine again passes through B.

If the triangle be equilateral, the curve degenerates into the three straight lines AO, BO, CO.

Note.— Join HA, HB, HC; then *both* O and I will be found in *one* of the triangles HAB, HBC, HCA. Since the curve can only cross the

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lines HA, HB, HC at H or at A, B, C, the serpentine branch must cut the base of the triangle in which O and I lie.

[Quest. 14975 (H. Bateman) may be treated on the same lines, its equation being $x(y^2-z^2)\sin A + \ldots = 0$.]

[Mr. R. Tucker, M.A., calls attention to a Question almost identical with Quest. 14975 set by Mr. F. D. Thomson and solved geometrically by Dr. Hirst (*v. Educational Times*, Oct., 1864, and *Reprint*, Vol. II., p. 57).—Editor.]

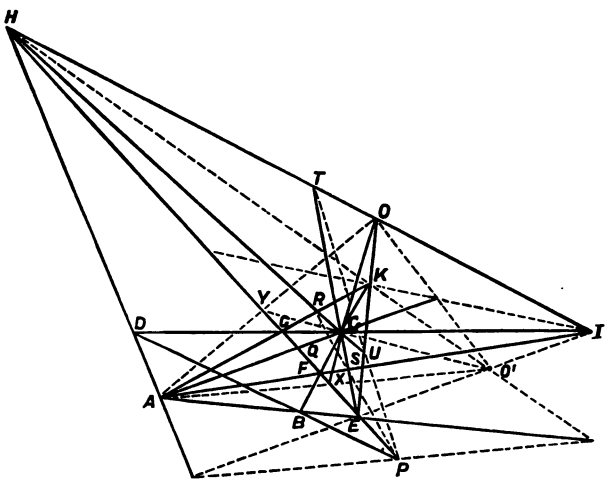
10049. (J. Lemaire.) — Soit un quadrilatère ABCD et une droite parallèle à la diagonale BD. Cette droite rencontre AB, BC, CD, DA respectivement en E, F, G, H ; AF et AG coupent CD et BC en I et K. Enfin EK et HI se rencontrent en O. Démontrer que la droite CO passe par le milieu de EH.

Solutions (I.) *by* C. E. Hillyer, M.A. ; (II.) *by* D. Biddle.

(I.) First, let EH be *any* transversal meeting BD in P and AC in Q.

Let OC meet EH in X ; then X shall be the harmonic conjugate of P with respect to E and H.

Let AK, CH meet in R, and AI, CE in S. Then, from the quadrilateral ABCD, we have P, E, F, Q, G, H in involution ; therefore P is also in the diagonal RS of the quadrilateral ASCR, *i.e.*, RS, EH, BD are concurrent.



Again, let CE, HI meet in T, and CH, EK in U. Then we have

$$(EFGH) = (I.EFGH) = (ESCT) ;$$

and again, $(EFGH) = (K.EFGH) = (UCRH) = (HRCU)$.

Therefore $$(ESCT) = (HRCU).$$

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Therefore TU passes through P. Therefore, from the quadrilateral CUOT, X is the harmonic conjugate of P with respect to E and H.

Now, when EH is parallel to BD, P is at an infinite distance, and X becomes the mid-point of EH.

Remarks.—The following are some additional points of interest in connexion with the figure:—

(1) If EI, HK meet in O′, OO′, AC, KI are concurrent in the harmonic conjugate of Q with respect to A and C.

(2) O′A passes through X, and OA, O′C also meet in a point Y in HE, and the intersection of OO′ and HE is the harmonic conjugate of Q with respect to X and Y.

(3) The intersection of HK with AB and the intersection of EI with AD are collinear with P.

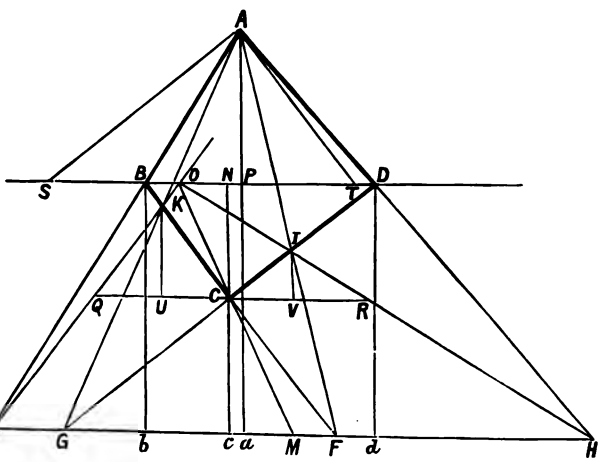
(4) If KI meets EH in V, then CV meets BD in the third diagonal of the quadrilateral ABCD.

(II.) This is equivalent to proving that, if through C QR be drawn parallel to EH, to meet EK in Q and HI in R, then CR = CQ. We will assume this equality. Having drawn certain perpendiculars to BD, QR, and EH, let $\angle BAP = \alpha$, $\angle BCN = \beta$, $\angle NCD = \gamma$, $\angle DAP = \delta$; also let QC = CR = x. Then we have

$$KU + Cc : EF = KU : x \quad \text{and} \quad IV + Cc : GH = IV : x;$$

therefore
$$\frac{GH}{EF} = \frac{1 + Cc/IV}{1 + Cc/KU};$$

and, since $Dd = Bb$,
$$\frac{GH}{EF} = \frac{\tan\gamma + \tan\delta}{\tan\alpha + \tan\beta}.$$



Now $IV + Cc = \dfrac{GF}{\tan\gamma + \tan FAa}$ and $KU + Cc = \dfrac{GF}{\tan\beta + \tan GAa}$;

therefore
$$IV = GF/(\tan\gamma + aF/Aa) - Cc$$

and
$$KU = GF/(\tan\beta + aG/Aa) - Cc.$$

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Thus we have

$$\frac{GH}{EF} = \frac{(IV + Cc)KU}{(KU + Cc)IV} = \frac{KU\,(Aa\tan\beta + aG)}{IV\,(Aa\tan\gamma + aF)} = \frac{Aa\,.\,GF - Cc\,(Aa\tan\beta + aG)}{Aa\,.\,GF - Cc\,(Aa\tan\gamma + aF)}.$$

But $$GF = Cc\,(\tan\beta + \tan\gamma).$$

Therefore $$\frac{GH}{EF} = \frac{\tan\gamma + \tan\delta}{\tan\alpha + \tan\beta} = \frac{Aa\,(\tan\beta + \tan\gamma) - (Aa\tan\beta + aG)}{Aa\,(\tan\beta + \tan\gamma) - (Aa\tan\gamma + aF)}$$
$$= \frac{Aa\tan\gamma - aG}{Aa\tan\beta - aF} = \frac{SD}{BT'}$$

as can be seen by drawing AS, AT parallel to DC, BC, respectively, and producing them to meet EH. Now

$$SD = AP\,(\tan\gamma + \tan\delta) \quad \text{and} \quad BT = AP\,(\tan\alpha + \tan\beta).$$

Hence CR = CQ, and OC produced bisects EH.

15001. (A. M. Nesbitt, M.A.)—Two conics U and V intersect in ABCO, and through O two arbitrary straight lines OP, OQ are drawn; OP cuts U in P and V in Q′; OQ cuts U in P′ and V in Q. Prove that the four straight lines OP, OQ, PP′, QQ′ and the three sides of the triangle ABC all touch one and the same conic.

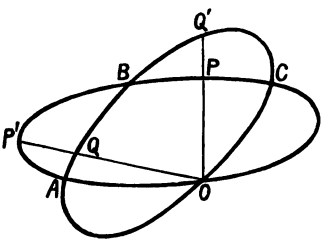
Solutions (I.) *by* R. F. Davis, M.A., *and the* Proposer;
(II.) *by* Professor Nanson; (III.) *by* Lionel E. Reay, B.A.

(I.) Projecting the figure so that B, C become the focoids, the theorem becomes:—If two circles U, V intersect in O, A, and through O two secants POQ′, QOP′ are drawn (P, P′ lying on U), then a parabola can be described with A as focus so as to touch POQ′, QOP′, PP′, and QQ′. This is obvious since the Simson lines of A with respect to the triangles OPP′, OQQ′ are coincident.

(II.) The envelope of a line cut in involution by three conics S_1, S_2, S_3 is a class cubic which when S_1, S_2, S_3 have a common point O reduces to the point O and a conic Σ. If, then, Δ_r be the triangle formed by the remaining intersections of S_p, S_q, where $p, q, r = 1, 2, 3$, the sides of the three triangles Δ_r touch Σ. Taking now S_1, S_2, S_3 to be U, V, and the line pair OP, OQ, the three triangles Δ_r are ABC, OPP′, OQQ′. Therefore, &c.

(III.) ABC and OPP′ are two triangles whose vertices lie on a conic; therefore their sides touch a conic, that is, PP′ touches the conic determined by the five tangents AB, BC, CA, OP, OQ.

Similarly, if *any* other conic through OABC cuts OP and OQ at Q′ and Q, then QQ′ touches this same conic.



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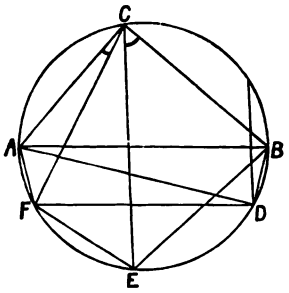
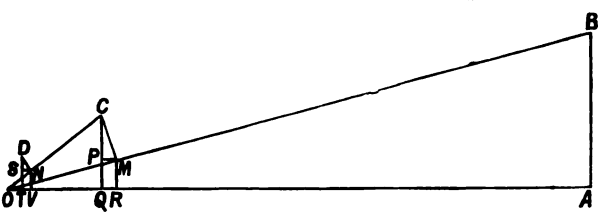
14079. (Rev. T. Roach, M.A.)—Prove geometrically
$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{1}{2}\pi$.

Solutions (I.) *by* R. Tucker, M.A., *and* Lionel E. Reay, B.A.;
(II.) *by the* Proposer.

(I.) Take a circle, diameter
AB = 65,
BC = BE = 52, AF = BD = 16;
then AE = 39, AD = BF = 63.
Now ABEF is a quadrilateral in the circle; therefore
AB . EF + AF . BE = AE . BF,
i.e., 65EF = 39 . 63 − 16 . 52,
or 5EF = 189 − 64 = 125;
therefore EF = 25;
hence right angle ACB = &c.



(II.) Take OA = 63, AB, perpendicular = 16; therefore OB = 65, $BOA=\sin^{-1}\frac{16}{65}$. In OB take OM = 12, MC, perpendicular = 5; therefore OC = 13, $COB=\sin^{-1}\frac{5}{13}$. Draw CPQ on MR perpendicular to OA;



then $CQ = CP + MR = 5\times\frac{63}{65}+12\times\frac{16}{65}=\frac{39}{5}$ and $OQ=\frac{52}{5}$. In OC take ON = 3, ND, perpendicular = 4; then OD = 5, $DOC=\sin^{-1}\frac{4}{5}$. If OD is not perpendicular to OA, let it take up some position as in figure, and draw DST and NV perpendicular to OA; then

$$DT = DS + NV = 4\times\frac{52}{5}+13+3\times\frac{39}{5}+13=5=OD;$$

therefore DOT must equal $\frac{1}{2}\pi$.

15089. (The late Professor R. B. Clayton.)—Normals lying in a given plane are drawn to a series of confocal quadrics. Show that they envelop a parabola.

Solution by the Proposer.

Since any normal is cut by the principal planes in a constant ratio, and the normals which lie in a given plane are met by the principal planes in three given lines, we have to find the envelope of a line meeting the sides of a fixed triangle in the points X, Y, Z, so that the ratio XY : XZ is constant. But this envelope is known to be a parabola.

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14922. (Robert W. D. Christie.)—Find a multiplier M which will make the modular equation $x^y - 1 = M(1/x^y - 1)$ true for any prime whatever, *e.g.*, $y = 6$, $x = 7$, $M = 8$, $7^6 - 1 = 8(\frac{1}{8} - 1)$, *i.e.*, $9 - 1 = \frac{8}{8} - 8$ or $-9 = 8 \bmod 17$.

Solution by the Proposer.

By the theory of primitive roots $r^{2m-1} + 1 = 0 \bmod P_{4m-1}$; therefore

$$r^{2m-3}(r^{2m-1} + 1) = 0; \quad \text{therefore} \quad -r^{2m-3} = r^{4m-4};$$

therefore
$$\frac{(r^{2m-3} - 1)(r^{2m-3})}{1 - r^{2m-3}} = r^{4m-4};$$

therefore
$$r^{2m-3} - 1 = r^{4m-4}\left(\frac{1 - r^{2m-3}}{r^{2m-3}}\right) = r^{2m-1} - r^{4m-4}$$
$$= r^{p-3}(r^{-(2m-3)} - 1),$$

whence $M = r^{p-3}$, where r is any root of the prime $p = 4m - 1$.

In an exactly similar way we establish

$$(r^{2m-2} - 1) = r^{p-3}(r^{-(2m-2)} - 1),$$

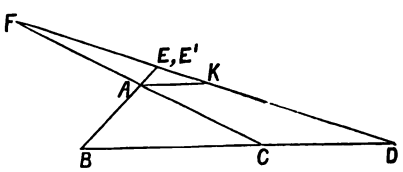
where r is any primitive root of the prime $p = 4m + 1$.

14888. (D. Biddle.)—ABC being a plane triangle having $a > b > c$, produce BC to D, BA to E, CA to F; and prove that, if $CD = ab/(a-b)$, $BE = ac/(a-c)$, and $AF = bc/(b-c)$, then D, E, F are collinear.

Solutions (I.) *by* H. W. Curjel, M.A., A. W. Poole, *and* H. Dew; (II.) *by* Rev. T. Wiggins, B.A.

(I.) $DB = a^2/(a-b)$, $AE = c^2/(a-c)$, $FC = b^2/(b-c)$; therefore $(EA \cdot FC \cdot DB)/(EB \cdot FA \cdot DC) = (a^2 \cdot b^2 \cdot c^2)/(ac \cdot bc \cdot ab) = 1$; therefore D, E, F are collinear.

(II.) Take AF and CD the given lengths. Produce BA to meet FD in E′. Then, if $AE' = c^2/(a-c)$, E′ coincides with E and F, E, D are collinear. Draw AK parallel to BD; therefore



$$FA/AK = FC/CD \quad \text{and} \quad AK/AE' = BD/BE';$$

therefore
$$FA/AE' = (FC \cdot BD)/(CD \cdot BE');$$

therefore
$$\frac{BE'}{AE'} = \frac{FC \cdot BD}{CD \cdot FA} = \frac{[b + bc/(b-c)][a + ab/(a-b)]}{[ab/(a-b)][bc/(b-c)]} = a/c;$$

therefore $(c + AE')/AE' = a/c$, whence $AE' = c^2/(a-c) = AE$.

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15101. (Professor Cochez.) — On donne une circonférence C, $x^2+y^2=R^2$, et la droite Δ $(x=a)$. On construit la circonférence C′ tangente à C et à Δ. La tangente commune T rencontre le diamètre C′A en M. Lieu de M.

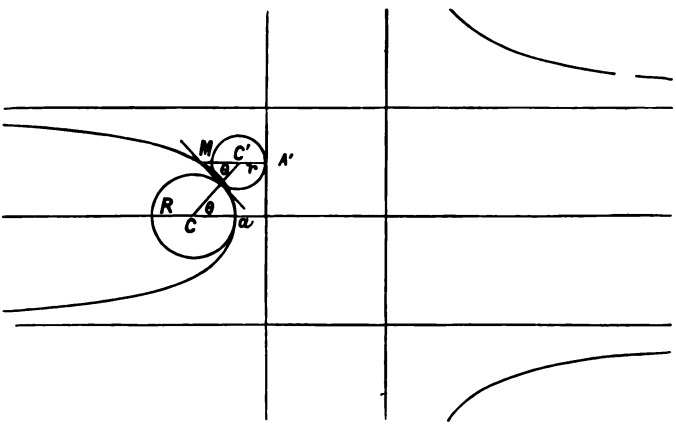
Solutions (I.) *by* J. Blaikie, M.A.; (II.) *by* Professor Sanjána, M.A.

(I.) Call CC′M θ, C′A r. Then

$$\cos\theta=(a-r)/(R+r),\quad \sin\theta=y/(R+r);$$

therefore $$r=(y^2-R^2+a^2)/\{2(R+a)\}.$$

Also $x=a-r-r/\cos\theta$; therefore $r=(a^2-ax)/(2a+R-x)$.



Equating values of r and simplifying, we get

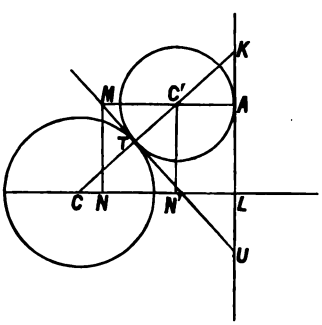
$$y^2(2a+R-x)=(R-x)(R+a)^2 \quad \text{or} \quad x-R=2ay^2/\{y^2-(R+a)^2\},$$

the equation to a cubic curve having three asymptotes $y=\pm(R+a)$ and $x=2a+R$.

(II.) Let $\angle KCL=\theta$; $MN=k$, $CN=h$; $CL=a$, $CT=R$.

The triangles UMA, UTK are evidently congruent; hence MA = TK. Thus

$$CN=CL-AM=CL-TK$$
$$=CL+CT-CK;$$
$$\therefore\quad h=a+R-a\sec\theta.$$



Also $C'T=C'A=C'K\cos\theta$
$$=(TK-C'T)\cos\theta;$$

hence $C'T=AM\cos\theta/(1+\cos\theta)$
$$=(a-h)/(1+\sec\theta);$$

$$\therefore\ k=MN=C'N'=(CT+C'T)\sin\theta=R\sin\theta+(a-h)\sin\theta/(1+\sec\theta)$$
$$=R\sin\theta+a(a-h)\sin\theta/(2a+R-h);$$

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i.e., $\sin\theta = k(2a+R-h)/[a(a-h)+R(2a+R-h)]$
$= k(2a+R-h)/[(a+R-h)(a+R)]$.

Eliminating θ, we get

$1 = \cos^2\theta + \sin^2\theta = a^2/(a+R-h)^2 + k^2(2a+R-h)^2/[(A+R-h)^2(a+R)^2]$;

$\therefore \quad (a+R-h)^2 - a^2 = k^2(2a+R-h)^2/(a+R)^2$;

$\therefore \quad (R-h)(a+R)^2 = k^2(2a+R-h)$.

Thus the locus is the cubic $y^2/(a+R)^2 = (R-x)/(2a+R-x)$.

When the contact is internal, we get

$$y^2/(a-R)^2 = (R+x)/(R+x-2a).$$

15099. (R. Knowles.)—From a point T (h, k) tangents are drawn to the parabola $y^2 = 4ax$; from a point P (x', y') a third tangent is drawn meeting these in M and N. Prove that the equation to the circle TMN is

$$x^2+y^2-\frac{ky'+2a(a+h)}{2a}x-\frac{2ak+y'(a-h)}{2a}y+\frac{ky'+2ah}{2}=0.$$

Solution by Professor Sanjána, M.A.

Assuming no geometrical properties of the tangent, we may proceed thus. Let the tangent at P (x', y') be $y = ma + a/m$; let the two tangents through T (h, k) be $y = m_{1,2}x + a/m_{1,2}$, so that $m_1+m_2 = k/h$, $m_1m_2 = a/h$. Then the three intersections are a/m_1m_2, $a(1/m_1+1/m_2)$; a/m_2m, $a(1/m_2+1/m)$; a/mm_1, $a(1/m+1/m_1)$. Assume equation of circle through these to be $x^2+y^2+Ax+By+C=0$, and find A, B, C. It will be found that

$$A = -a/m_1m_2m(m_1+m_2+m+m_1m_2m),$$

$$B = -a/m_1m_2m(m_1m_2+m_2m+mm_1-1), \quad C = a^2/m_1m_2m(m_1+m_2+m).$$

Substitute these values, and put $m = 2a/y'$; we get

$$(x^2+y^2)\frac{2a^2}{y'h}-a\left(\frac{k}{h}+\frac{2a}{y'}+\frac{2a^2}{y'h}\right)x-a\left(\frac{a}{h}+\frac{2ak}{y'h}-1\right)y+a^2\left(\frac{k}{h}+\frac{2a}{y'}\right)=0,$$

whence we derive

$$x^2+y^2-(ky'+2ah+2a^2)\frac{x}{2a}-(2ak+ay'-y'h)\frac{y}{2a}+\frac{ky'+2ah}{2}=0.$$

Of course the point $(a, 0)$ lies on this circle.

14991. (Professor Umes Chandra Ghosh.)—If $f(r_1r_2) = 0$ be the equation of a plane curve, where r_1 and r_2 are the distances of any point on the curve from two fixed points S_1, S_2 in its plane, and if ρ_1, ρ_2 be the distances of any point on a normal to the curve from S_1, S_2, show that

$$\left[\frac{\rho_1^2-\rho_2^2-(a_1^2-a_2^2)}{r_1^2-r_2^2-(a_1^2-a_2^2)}\right]^2 = \frac{a(\rho_1^2-a_1^2)+a_1[(a_1^2-a_2^2)-(\rho_1^2-\rho_2^2)]}{a(r_1^2-a_1^2)+a_1[(a_1^2-a_2^2)-(r_1^2-r_2^2)]},$$

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where $a_1 = a\dfrac{r_1(\partial f/\partial r_2)}{r_1(\partial f/\partial r_2)+r_2(\partial f/\partial r_1)}$, $a_2 = a\dfrac{r_2(\partial f/\partial r_1)}{r_1(\partial f/\partial r_2)+r_2(\partial f/\partial r_1)}$,

and $S_1S_2 = a$.

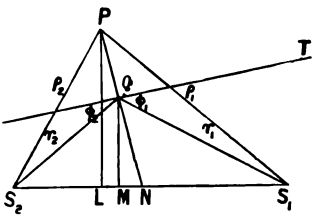
Solution by A. W. Poole.

$dr_1/ds = \cos\phi_1$, $dr_2/ds = \cos\phi_2$,

$f(r_1r_2) = 0$;

$\therefore\ (\partial f/\partial r_1)\,dr_1 + (\partial f/\partial r_2)\,dr_2 = 0$;

$\therefore\ (\partial f/\partial r_1)\sin \mathrm{NQS}_1 = (\partial f/\partial r_2)\sin \mathrm{NQS}_2$.



Also $\mathrm{NS}_1 \sin \mathrm{N} = r_1 \sin \mathrm{NQS}_1$,

$\mathrm{NS}_2 \sin \mathrm{N} = r_2 \sin \mathrm{NQS}_2$;

therefore $\dfrac{\mathrm{NS}_1}{\mathrm{NS}_2} = \dfrac{r_1 \sin \mathrm{NQS}_1}{r_2 \sin \mathrm{NQS}_2} = \dfrac{r_1(\partial f/\partial r_2)}{r_2(\partial f/\partial r_1)}$;

therefore, since $S_1S_2 = a$, $\mathrm{NS}_1 = a\dfrac{r_1(\partial f/\partial r_2)}{r_1(\partial f/\partial r_2)+r_2(\partial f/\partial r_1)}$, ...;

therefore $\mathrm{NS}_1 = a_1$, $\mathrm{NS}_2 = a_2$;

therefore $\mathrm{QN}^2 = r_1^2 + \mathrm{NS}_1^2 - 2r_1 \cdot \mathrm{NS}_1 \cos \mathrm{QS}_1\mathrm{N}$

$$= a_1^2 + r_1^2 - 2a_1r_1\left[\frac{a^2+r_1^2-r_2^2}{2ar_1}\right]$$

$$= a_1^2 + r_1^2\left[1-\frac{a_1}{a}\right] - a_1a + r_2^2\frac{a_1}{a} = \frac{a_2r_1^2}{a} + \frac{a_1r_2^2}{a} - a_1(a-a_1)$$

$$= (a_2r_1^2 + a_1r_2^2 - aa_1a_2)/a,$$

also $\mathrm{PN}^2 = (a_2\rho_1^2 + a_1\rho_2^2 - aa_1a_2)/a$

and $a(\rho_1^2 - a_1^2) + a_1[a_1^2 - a_2^2 - (\rho_1^2 - \rho_2^2)]$

$$= \rho_1^2(a-a_1) + \rho_2^2a_1 + a_1[a_1^2 - a_2^2 - aa_1]$$

$$= a_2\rho_1^2 + a_1\rho_2^2 + a_1[a(a_1-a_2) - aa_1] = a_2\rho_1^2 + a_1\rho_2^2 - aa_1a_2;$$

therefore right-hand expression $= (\mathrm{PN})^2/(\mathrm{QN})^2$.

From P and Q drop perpendiculars PL, QM on S_1S_2; therefore

$$\rho_1^2 - \rho_2^2 - (a_1^2 - a_2^2) = S_1L^2 - S_2L^2 - (S_1N^2 - S_2N^2)$$

$$= S_1L^2 - S_1N^2 + (S_2N^2 - S_2L^2)$$

$$= \mathrm{LN}^2 + 2\mathrm{LN} \cdot \mathrm{NS}_1 + \mathrm{LN}^2 + 2S_2\mathrm{L} \cdot \mathrm{LN}$$

$$= 2\mathrm{LN}[\mathrm{LN} + \mathrm{NS}_1 + S_2\mathrm{L}] = 2\mathrm{LN} \cdot a;$$

therefore left-hand expression $= \mathrm{LN}^2/\mathrm{MN}^2$, which $= \mathrm{PN}^2/\mathrm{QN}^2 =$ right-hand expression.

14849. (Lt.-Col. Allan Cunningham, R.E.)—Resolve into *prime* factors $\mathrm{N} = (11^{33}+1)$.

Solution by the Proposer.

$$\mathrm{N} = x^{33}+y^{33} = \left(\frac{x^{33}+y^{33}}{x^3+y^3}\,\frac{x+y}{x^{11}+y^{11}}\right)\frac{x^{11}+y^{11}}{x+y}\,\frac{x^3+y^3}{x+y}(x+y).$$

Let $\mathrm{F}_1 = (x+y)$, $\mathrm{F}_3 = \dfrac{x^3+y^3}{x+y}$, $\mathrm{F}_{11} = \dfrac{x^{11}+y^{11}}{x+y}$, $\mathrm{F}'_{33} = \dfrac{x^{33}+y^{33}}{x^3+y^3}$.

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Then $$N = F'_{33}/F_{11} \cdot F_{11} \cdot F_3 \cdot F_1.$$

Let $x = 1$, $y = 11$; then $F_1 = 4 \cdot 3$, $F_3 = 111 = 3 \cdot 7$. Also

$$F_{11} = \left\{(x+y)^5 - 11x^2y^2(x+y)\right\}^2 - 11xy\left\{\frac{x^5-y^5}{x-y} - 2x^2y^2\right\}^2,$$

as may be verified by actual expansion. Here F_{11} may be resolved* into two factors $F_{11} = L_{11}, M_{11}$, which are given by

$$L_{11} \text{ or } M_{11} = \left\{(x+y)^5 - 11x^2y^2(x+y)\right\} \mp \sqrt{(11xy)}\left\{\frac{x^5-y^5}{x-y} - 2x^2y^2\right\},$$

the expressions for L_{11}, M_{11} differing only in the sign ($\mp$) before $\sqrt{(11xy)}$.

Now, let $x = 1$, $y = 11$; then $\sqrt{(11xy)} = 11$, and L, M are *both rational*.

Here $$L_{11} \text{ or } M_{11} = (12^5 - 11^3 \cdot 12) \mp 11\left(\frac{11^5-1}{11-1} - 2 \cdot 11^2\right)$$
$$= 248832 \mp 11 \cdot 15863;$$

therefore $$L_{11} = 58367, \quad M_{11} = 407353 = 23 \cdot 89 \cdot 199;$$

and $$F_{11} = L_{11} \cdot M_{11}.$$

Next, let $x = 1^3$, $y = 11^3 = 1331$; then $\sqrt{(11xy)} = 121$; and F'_{33} may be resolved by the same rule as F_{11}, so that $F'_{33} = L_{33} \cdot M_{33}$, where

$$L_{33} \text{ or } M_{33} = (1332^5 - 11 \cdot 1331^2 \cdot 1332) \mp 121\left(\frac{1331^5-1}{1331-1} - 2 \cdot 1331^2\right)$$
$$= 4192,937,951,586,660 \mp 121 \cdot 3140,784,554,183,$$

wherein L_{33}, M_{33} differ only by the ($\mp$) sign before $\sqrt{(11xy)}$. Hence

$$L_{33} = 3812,903,020,530,517; \quad M_{33} = 4572,972,882,642,803.$$

Here $F'_{33} = L_{33} \cdot M_{33}$ is divisible by $F_{11} = L_{11} \cdot M_{11}$; and, by the properties of such numbers, L_{33} contains L_{11} or M_{11}, and M_{33} contains M_{11} or L_{11}. On actual trial division, it is found that

$$L_{33}/M_{11} = 9360,193,789; \quad M_{33}/L_{11} = 78,348,602,509.$$

Next, a MS. table of roots of the congruence $\frac{y^{11}+1}{y+1} \equiv 0 \pmod{p}$, which the Proposer has compiled and hopes to publish shortly, shows that 67 and 9439 are divisors of $\left(\frac{11^{33}+1}{11^3+1} \cdot \frac{11+1}{11^{11}+1}\right)$, and that there are no more divisors < 10000. On actual division, it is found that

$$L_{33}/M_{11} = 9439 \cdot 991,651; \quad M_{33}/L_{11} = 67 \cdot 1169,382,127.$$

The application of Laurence's process (*Proc. Lond. Math. Soc.*, Vol. xxviii., "On the Determination of certain Primes") shows that the last large factor is a *prime*. Hence N is *completely* resolved in form

$$N = (L_{33}/M_{11})(M_{33}/L_{11})(L_{11} \cdot M_{11})\, F_3 \cdot F_1.$$

12685. (R. Tucker, M.A.)—P(m_1), Q(m_2), R(m_3) are conormal points on a parabola, and pqr is the tangent triangle. PP_q, QQ_p, parallel

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* This resolution of F_{11} into the form $F_{11} = P^2 - 11xy \cdot Q^2$, which becomes a difference of squares when $x = 1$, $y = 11$, 11^3, 11^5, &c., is due to the late M. Aurifeuille (see Ed. Lucas's paper "Sur la Série récurrente de Fermat," Rome, 1879, p. 6). Such formulæ have been called by the Proposer *Aurifeuillians*.

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to the tangents at Q, P, respectively, meet the curve in P_q, Q and cut one another in R_r; P_p, Q_q are points obtained in a similar manner. PP_p, QQ_q meet in r_3, and so on; prove that

$$xy = -m_1m_2m_3,\ a^2 \quad \text{.....(i.)}$$

is a hyperbola circumscribing the triangles pqr, $p_2q_2r_2$, and that the orthocentres of these triangles coincide, and, of course, lie on (i.). The circumcircles of these triangles cut the latus rectum in the same point, the join of which to the orthocentre is a diameter of (i.). The parallels through P, Q to the normals at Q, P intersect in r, and so on; then the orthocentre of $p_1q_1r_1$ is on (i.), as is also the centre of perspective of the triangles PQR, pqr.

Solution by W. J. Greenstreet, M.A.

We easily find the following coordinates:—

p, $(am_2m_3, -am_1)$, &c.; therefore p, q, r lie on $xy = -a^2m_1m_2m_3$.....(1),

p_2, $(am_1^2, -am_2m_3/m_1)$, &c.; therefore p_2, q_2, r_2 lie on (1). The orthocentre of $p_2q_2r_2$ is $(-a, am_1m_2m_3)$, and from the properties of the hyperbola will with that of pqr lie on (1).

The circles pqr, $p_2q_2r_2$ are

$$x^2+y^2-(8-\tfrac{1}{2}\Sigma m_1^2)\,ax+\tfrac{1}{2}am_1m_2m_3y+\tfrac{1}{2}a^2\,(m_1^2m_2^2m_3^2-2\Sigma m_1^2+8)=0\ldots(2),$$

$$x^2+y^2-(1+\Sigma m_1^2)\,ax+[m_1m_2m_3+(\Sigma m_1^2/4m_1m_2m_3)]\,ay + \tfrac{1}{4}a^2\,[4\Sigma m_1^2+(\Sigma m_1^2)^2]=0\ldots\ldots(3).$$

These cut the latus rectum in $(a, -am_1m_2m_3)$. The join of this to $(-a, am_1m_2m_3)$ is obviously a diameter as stated. Pr_1 is

$$y+m_2x=am_1\,(2+m_1m_2),$$

r_1 is $[-a\,(2+m_1m_2),\ -a\,(2m_3+m_1m_2m_3)]$. The orthocentre of (p_1, q_1, r_1) is $(2a, -\frac{1}{2}am_1m_2m_3)]$, and is therefore on (1). The centre of perspective in the question is $(\frac{1}{2}a\Sigma m_1^2, -6am_1m_2m_3/\Sigma m_1^2)$, which is also on (1).

10097. (W. J. C. Sharp, M.A.)—Prove that the feet of the perpendiculars from any point, whose isogonal conjugate lies upon the plane at infinity, upon the faces of the tetrahedron of reference lie in a plane.

Solution by R. F. Whitehead.

Since the point is the isogonal conjugate of a point at infinity, it is the focus of a paraboloid of revolution touching the faces of the tetrahedron. Hence the perpendiculars from the focus on the faces meet them in points on the plane which touches the paraboloid at its vertex.

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